İzmir Institute of Technology
Math 255 Differential Equations, Summer 2023
Final Examination

Name: ______________________________
Student ID: ______________________________
Department: ______________________________

Duration: 150 Minutes

This exam contains 12 pages (check), including this page. Organize your work in the space provided. You will be provided Laplace transform table sheet.

- You may not use books, notes or any calculator.
- A correct answer presented without any calculation will receive no credit.
- A correct answer without any explanations will not recieve full credit. You are expected to clarify/explain your work as much as you can.
- An incorrect answer including partially correct calculations/explanations will receive partial credit.
- You are expected justify your claims unless you are using results from the lecture. Claims without any clarification will not be scored.

Grade Table

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	20	20	20	120
Score:							

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1. Consider the differential equation

$$\frac{dy}{dx} = -\frac{x + 3x^3 \sin y}{x^4 \cos y}. \qquad (1)$$

(a) (4 points) Show that equation (1) is not exact.

$$\frac{dy}{dx} = -\frac{x + 3x^3 \sin y}{x^4 \cos y}$$

$$\Rightarrow \underbrace{(x + 3x^3 \sin y)}_{M(x,y)}dx + \underbrace{x^4 \cos y}_{N(x,y)}dy = 0 \quad (*)$$

$$\frac{\partial}{\partial y}M(x,y) = 3x^3 \cos y$$

$$\frac{\partial}{\partial x}N(x,y) = 4x^3 \cos y$$

$$\frac{\partial}{\partial y}M(x,y) \neq \frac{\partial}{\partial x}N(x,y)$$

$\Rightarrow$ (1) is not exact.

(b) (4 points) Find an integration factor of x, i.e., $\mu = \mu(x)$, so that (1) becomes an exact differential equation when multiplied by $\mu(x)$.

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3x^3\cos y - 4x^3\cos y}{x^4\cos y} = \frac{-x^3\cos y}{x^4\cos y} = -\frac{1}{x}$$

is a function of x. Therefore an integration factor for (1) is

$$\mu(x) = e^{\int\left(-\frac{1}{x}\right)dx} = e^{-\ln x} = \frac{1}{x}.$$

(c) (12 points) Following from part (b), find the general solution.

Multiplying (*) by $\mu(x) = \frac{1}{x}$, we get

$$\underbrace{(1 + 3x^2\sin y)}_{\tilde{M}(x,y)}dx + \underbrace{x^3\cos y}_{\tilde{N}(x,y)}\,dy = 0$$

* Integrate $\tilde{M}(x,y)$ with respect to x:

$$\int \tilde{M}(x,y)\,dx = \int(1 + 3x^2\sin y)\,dx = x + x^3\sin y + g(y).$$

* Differentiate the result with respect to y:

$$\frac{\partial}{\partial y}\left(x + x^3\sin y + g(y)\right) = x^3\cos y + g'(y).$$

* This must be equal to $\tilde{N}(x,y)$:

$$x^3\cos y + g'(y) = \tilde{N}(x,y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = c.$$

So the general solution is

$$x + x^3\sin y + c = 0.$$

2. Consider the second order Cauchy–Euler equation

$$x^2y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0. \tag{2}$$

(a) (7 points) Find the fundamental set of solutions $\{y_1(x), y_2(x)\}$ of the homogeneous part.

Homogeneous part: $x^2y'' - 3xy' + 4y = 0$. (*)

Let $y = x^r$. Then $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$. We substitute into (*) to get

$$r(r-1) - 3r - 4 = 0 \Rightarrow r_{1,2} = 2$$

$$\Rightarrow y_1(x) = x^2, \quad y_2(x) = x^2 \ln x.$$

So fundamental set of solutions is:

$$\{x^2, x^2 \ln x\}.$$

(b) (13 points) Find the solution of (2) subject to the initial conditions $y(1) = 1$, $y'(1) = 3$.

We divide the equation by x^2 to write it in standard form as

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = \ln x \Rightarrow g(x) = \ln x$$

We apply the Method of Variation of Parameters to obtain a particular solution.

Particular Solution

$$W(y_1, y_2)(x) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x\ln x + x \end{vmatrix} = x^3$$

$$u_1(x) = -\int \frac{g(x)\,y_2(x)}{W(y_1,y_2)(x)}\,dx = -\int \frac{\ln x \; x^2 \ln x}{x^3}\,dx = -\int \frac{(\ln x)^2}{x}\,dx$$

$t = \ln x$, $dt = \frac{1}{x}\,dx$

$$= -\int t^2\,dt = -\frac{(\ln x)^3}{3}$$

$$u_2(x) = \int \frac{g(x)\,y_1(x)}{W(y_1,y_2)(x)}\,dx = \int \frac{\ln x \; x^2}{x^3}\,dx = \int \frac{\ln x}{x}\,dx$$

$t = \ln x$, $dt = \frac{1}{x}\,dx$

$$= \int t\,dt = \frac{(\ln x)^2}{2}$$

So particular solution is

$$Y(x) = y_1 u_1 + y_2 u_2 = -x^2\frac{(\ln x)^3}{3} + x^2 \ln x \cdot \frac{(\ln x)^2}{2} = \frac{1}{6}x^2(\ln x)^3.$$

<u>General solution</u>

$$y_g(x) = c_1 y_1(x) + c_2 y_2(x) + Y(x) = c_1 x^2 + c_2 x^2 \ln x + \frac{1}{6}x^2(\ln x)^3.$$

<u>Solution of initial value problem</u>

Differentiating $y_g(x)$

$$y_g'(x) = 2c_1 x + 2c_2 x \ln x + c_2 x + \frac{1}{3}x(\ln x)^3 + \frac{1}{2}x(\ln x)^2.$$

Now we employ initial conditions to get

$$x = 1 \Rightarrow \begin{cases} y(1) = c_1 = 1 \\ y'(1) = 2c_1 + c_2 = 3 \end{cases} \Rightarrow c_1 = 1,\ c_2 = 1.$$

Hence

$$y(x) = x^2 + x^2 \ln x + \frac{1}{6}x^2(\ln x)^3$$

3. Consider the forth order differential equation

$$y^{(4)} + y = e^{\frac{\sqrt{2}t}{2}} \sin\left(\frac{\sqrt{2}}{2}t\right). \tag{3}$$

(a) (10 points) Find the solution of the homogeneous part.

Characteristic equation is

$$m^4 + 1 = 0 \implies m^4 = -1 = e^{i\pi}.$$

We seek forth order roots of $e^{i\pi}$ in set of complex numbers.

$$m^4 = e^{i\pi} = e^{i(\pi + k\cdot 2\pi)} \implies m = e^{i\left(\frac{\pi}{4} + k\frac{\pi}{2}\right)}, \quad k \in \mathbb{Z}.$$

$$k=0: \quad m_1 = e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$k=1: \quad m_2 = e^{i\frac{3\pi}{4}} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$k=2: \quad m_3 = e^{i5\pi/4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$k=3: \quad m_4 = e^{i7\pi/4} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

Hence

$$y_h(t) = e^{\frac{\sqrt{2}}{2}t}\Big(c_1\cos\big(\tfrac{\sqrt{2}}{2}t\big) + c_2\sin\big(\tfrac{\sqrt{2}}{2}t\big)\Big) + e^{-\frac{\sqrt{2}}{2}t}\Big(c_3\cos\big(\tfrac{\sqrt{2}}{2}t\big) + c_4\sin\big(\tfrac{\sqrt{2}}{2}t\big)\Big)$$

$$1 \qquad 2 \qquad 3 \qquad 4$$

(b) (10 points) Write the general solution of (3) without computing the numerical coefficients.

$$g(t) = e^{\frac{\sqrt{2}}{2}t}\sin\left(\tfrac{\sqrt{2}}{2}t\right) \implies Y(t) = e^{\frac{\sqrt{2}}{2}t}\left(A\sin\left(\tfrac{\sqrt{2}}{2}t\right) + B\cos\left(\tfrac{\sqrt{2}}{2}t\right)\right)$$

Method of Undetermined Coefficients

Observe that $Y(t)$ involves linearly dependent parts with 1 and 2 of the homogeneous solution. So we update $Y(t)$ as

$$\tilde{Y}(t) = t e^{\frac{\sqrt{2}}{2}t}\left(A\sin\left(\tfrac{\sqrt{2}}{2}t\right) + B\cos\left(\tfrac{\sqrt{2}}{2}t\right)\right).$$

Consequently general solution is

$$y_g(x) = y_h(t) + \tilde{Y}(t).$$

4. Find the power series solution of the differential equation

$$y'' - 4xy' - 8y = 0$$

in terms of powers of x, i.e., about $x_0 = 0$ by following the steps below:

(a) (10 points) Determine the recurrence relation.

We seek the general solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} a_n n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}.$$

So we substitute these into the equation and get

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - 4x \sum_{n=1}^{\infty} a_n n x^{n-1} - 8 \sum_{n=0}^{\infty} a_n x^n = 0$$

All powers to x^n

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) x^n - 4 \sum_{n=1}^{\infty} a_n n x^n - 8 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow 2a_2 - 8a_0 + \sum_{n=1}^{\infty} \left[a_{n+2}(n+2)(n+1) - 4na_n - 8a_n \right] x^n = 0$$

$$\Rightarrow a_2 = 4a_0, \quad a_{n+2} = \frac{4a_n(2+n)}{(n+2)(n+1)}, \quad n = 1, 2, \dots$$

$$= \frac{4a_n}{n+1}$$

(b) (10 points) Find first 3 nonzero terms for each linearly independent solution.

$n = 1 \Rightarrow a_3 = \dfrac{4a_1}{(1+1)} = 2a_1$

$n = 2 \Rightarrow a_4 = \dfrac{4a_2}{(2+1)} = \dfrac{16}{3}a_0$

$n = 3 \Rightarrow a_5 = \dfrac{4a_3}{(3+1)} = 2a_1$

$\vdots$

$$y_g(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0\left(1 + 4x^2 + \frac{16}{3}x^4 + \cdots\right) + a_1\left(x + 2x^3 + 2x^5 + \cdots\right)$$

5. (20 points) Use the Laplace transform to solve the initial value problem

$$\begin{cases} y'' + 2y' + 2y = g(t), \\ y(0) = 0, \quad y'(0) = 2, \end{cases}$$

where

$$g(t) = \begin{cases} 1, & \pi \le t < 2\pi, \\ 0, & \text{elsewhere.} \end{cases}$$

$$g(t) = \begin{cases} 1, & \pi \le t < 2\pi \\ 0, & \text{elsewhere} \end{cases} = u_{\pi}(t) - u_{2\pi}(t).$$

We apply the Laplace transform and get

$$s^2 Y(s) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=2} + 2s\,Y(s) - 2\underbrace{y(0)}_{=0} + 2\,Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}$$

$$\Rightarrow Y(s)\left(s^2+2s+2\right) = 2 + \frac{1}{s}e^{-\pi s} - \frac{1}{s}e^{-2\pi s}$$

$$\Rightarrow Y(s) = \underset{(1)}{\frac{2}{s^2+2s+2}} + \underset{(2)}{\frac{1}{s(s^2+2s+2)}}\,e^{-\pi s} - \underset{(3)}{\frac{1}{s(s^2+2s+2)}}\,e^{-\pi s}$$

To apply inverse Laplace transform let us first decompose (2) (and (3)) into fractions with irreducible denominators.

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2} \Rightarrow As^2 + 2As + 2A + Bs^2 + Cs = 1$$

$$\Rightarrow (A+B)s^2 + (2A+C)s + 2A = 1$$

$$\Rightarrow A = 1/2, \quad B = -\frac{1}{2}, \quad C = -1.$$

So we rewrite $Y(s)$ as

$$Y(s) = 2\cdot\frac{1}{(s+1)^2+1} + \left(\frac{1}{2}\,\frac{1}{s} - \frac{1}{2}\,\frac{s+2}{(s+1)^2+1}\right)e^{-\pi s} - \left(\frac{1}{2}\,\frac{1}{s} - \frac{1}{2}\,\frac{s+2}{(s+1)^2+1}\right)e^{-2\pi s}$$

$$\frac{s+1}{(s+1)^2+1} \qquad \frac{1}{(s+1)^2+1} \qquad\qquad \frac{s+1}{(s+1)^2+1} \qquad \frac{1}{(s+1)^2+1}$$

$$= 2\cdot\frac{1}{(s+1)^2+1} \longrightarrow Y_1(s)$$

$$+\frac{1}{2}\left(\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right)e^{-\pi s} \longrightarrow Y_2(s)$$

$$-\frac{1}{2}\left(\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right)e^{-2\pi s} \longrightarrow Y_3(s)$$

We apply inverse Laplace transform to get

$$\mathcal{L}^{-1}\{Y_1(s)\} = 2e^{-t}\sin t$$

$$\mathcal{L}^{-1}\{Y_2(s)\} = \frac{1}{2}u_\pi(t)\left(1 - e^{-(t-\pi)}\cos(t-\pi) + e^{-(t-\pi)}\sin(t-\pi)\right)$$

$$= \frac{1}{2}u_\pi(t)\left(1 + e^{-t+\pi}\cos t - e^{-t+\pi}\sin t\right)$$

$$\mathcal{L}^{-1}\{Y_3(s)\} = \frac{1}{2}u_{2\pi}(t)\left(1 - e^{-(t-2\pi)}\cos(t-2\pi) + e^{-(t-2\pi)}\sin(t-2\pi)\right)$$

$$= \frac{1}{2}u_{2\pi}(t)\left(1 - e^{-t+2\pi}\cos t + e^{-t+2\pi}\sin t\right)$$

Hence

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\{Y_1(s)\} + \mathcal{L}^{-1}\{Y_2(s)\} + \mathcal{L}^{-1}\{Y_3(s)\}$$

$$= 2e^{-t}\sin t + \frac{1}{2}u_\pi(t)\left(1 + e^{-t+\pi}\cos t - e^{-t+\pi}\sin t\right)$$

$$- \frac{1}{2}u_{2\pi}(t)\left(1 - e^{-t+2\pi}\cos t + e^{-t+2\pi}\sin t\right).$$

6. Consider the following system of first order linear differential equations.

$$\begin{cases} y_1' = 5y_1 + 3y_2, \\ y_2' = -y_1 + y_2. \end{cases} \tag{4}$$

(a) (7 points) Find eigenvalue(s) and associated eigenvectors of the coefficients matrix.

Coefficients matrix is $A = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix}$. Associated characteristic polynomial is

$$p_\lambda(A) = \begin{vmatrix} 5-\lambda & 3 \\ -1 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda)+3 = \lambda^2 - 6\lambda + 8$$

$$p_\lambda(A) = 0 \iff \lambda^2 - 6\lambda + 8 = 0 \iff \lambda_1 = 2, \quad \lambda_2 = 4$$

$\underline{\lambda_1 = 2}$: $3v_1^{(1)} + 3v_2^{(1)} = 0$

$\Rightarrow v_1^{(1)} = s, \quad v_2^{(1)} = -s, \quad s \in \mathbb{R}.$

Let $s=1$. Then

$$v^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\underline{\lambda_2 = 4}$: $v_1^{(2)} + 3v_2^{(2)} = 0$

$\Rightarrow v_1^{(2)} = 3s, \quad v_2^{(2)} = -s, \quad s \in \mathbb{R}$

Let $s=1$. Then

$$v^{(2)} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

(b) (7 points) Find the fundamental matrix, $\Phi(t)$, with the property that $\Phi(0) = \mathbf{I}$. Here $\mathbf{I}$ represents the 2−dimensional identity matrix.

Fundamental set of solution is $\left\{ \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}, \begin{pmatrix} 3e^{4t} \\ -e^{4t} \end{pmatrix} \right\}$, so

$$\Psi(t) = \begin{pmatrix} e^{2t} & 3e^{4t} \\ -e^{2t} & -e^{4t} \end{pmatrix} \Rightarrow \Psi(0) = \begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix}$$

$$\Rightarrow \Psi^{-1}(0) = \frac{1}{-1+3} \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1/2 & -3/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Hence

$$\Phi(t) = \Psi(t)\Psi^{-1}(0) = \begin{pmatrix} e^{2t} & 3e^{4t} \\ -e^{2t} & -e^{4t} \end{pmatrix} \begin{pmatrix} -1/2 & -3/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{e^{2t}}{2} + \frac{3e^{4t}}{2} & -\frac{3e^{2t}}{2} + \frac{3e^{4t}}{2} \\ \frac{e^{2t}}{2} - \frac{e^{4t}}{2} & \frac{3e^{2t}}{2} - \frac{e^{4t}}{2} \end{pmatrix}$$

(c) (6 points) By using the fundamental matrix $\mathbf{\Phi}(t)$ you found in part (b), find the solution of (4) subject to the initial condition

$$\mathbf{y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (\star)$$

Solution of (4) subject to the initial conditions $(\star)$ is

$$y = \begin{pmatrix} -\frac{e^{2t}}{2} + \frac{3e^{4t}}{2} & -\frac{3e^{2t}}{2} + \frac{3e^{4t}}{2} \\ \frac{e^{2t}}{2} - \frac{e^{4t}}{2} & \frac{3e^{2t}}{2} - \frac{e^{4t}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{e^{2t}}{2} + \frac{3e^{4t}}{2} + \frac{3e^{2t}}{2} - \frac{3e^{4t}}{2} \\ \frac{e^{2t}}{2} - \frac{e^{4t}}{2} - \frac{3e^{2t}}{2} + \frac{e^{4t}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}$$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\frac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\frac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. t^n; $n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 6
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}$, $n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)\,f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}\,f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 19
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28