03.08.2023

İzmir Institute of Technology
Math 255 Differential Equations, Summer 2023
Midterm Examination I

Name: Solution Key
Student ID:
Department:

Duration: 105 Minutes

This exam contains 6 pages (check), including this page. Organize your work in the space provided. If necessary, you will be provided one empty sheet.

- You may not use books, notes or any calculator.
- A correct answer presented without any calculation will receive no credit.
- A correct answer without any explanations will not recieve full credit. You are expected to clarify/explain your work as much as you can.
- An incorrect answer including partially correct calculations/explanations will receive partial credit.
- You are expected justify your claims unless you are using results from the lecture. Claims without any clarification will not be scored.

Grade Table

Question:	1	2	3	4	Total
Points:	20	25	30	25	100
Score:					

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1. (20 points) If the Wronskian $W(f,g)$ of $f(t)$ and $g(t)$ is $2t^4\cos t$, and if $f(t) = t^2$, find $g(t)$.

$$W(f,g) = \begin{vmatrix} t^2 & g \\ 2t & g' \end{vmatrix} = 2t^4\cos t$$

$$\Rightarrow g't^2 - 2tg = 2t^4\cos t$$

$$\Rightarrow g' - \frac{2}{t}g = 2t^2\cos t \rightarrow 1^{st} \text{ order, linear}$$

Integration factor: $\mu(t) = e^{\int -\frac{2}{t}dt} = e^{-2\ln t} = t^{-2}$

$\Rightarrow t^{-2}y' - 2t^{-3}y = 2\cos t$

$\Rightarrow (y t^{-2})' = 2\cos t$

$\Rightarrow y t^{-2} = 2\sin t \Rightarrow y(t) = 2t^2 \sin t$

2. Consider the logistic equation

$$\begin{cases} y' = y\left(1 - \dfrac{y}{K}\right), & t > 0, \\ y(0) = y_0, \end{cases} \tag{1}$$

where t is the independent variable that represents time, $K > 0$ is the carrying capacity and y_0 is the amount of population at $t = 0$.

(a) (5 points) Classify the differential equation. According to your classification, explain your solution strategy. Give details as much as you can.

2. 1st order, nonlinear, Bernoulli, with $n=2$.

Strategy. $z = y^{1-2} \Rightarrow z = y^{-1} \Rightarrow z' = -y^{-2}y'$

(b) (5 points) Is there any equilibrium solutions? Explain.

We seek a solution so that it remains unchanged as t varies, i.e., $y' = 0$.

$$y' = 0 \Rightarrow y\left(1 - \frac{y}{K}\right) = 0$$
$$\Rightarrow y = 0 \text{ or } y = K.$$

(c) (15 points) Let $y_0 > 0$. Solve the initial value problem (1). Then show that all solutions approach to the capacity as time tends to infinity, i.e., for all $y_0 > 0$ show that

$$\lim_{t\to\infty} y(t) = K.$$

Let us express the eqn. as

$$-y' - y = -\frac{y^2}{K}$$

$z = y^{-1}$
$z' = -y^{-2}y'$

$$e^t / z' + z = \frac{1}{K}$$

$$\Rightarrow (ze^t)' = \frac{e^t}{K}$$

$$\Rightarrow ze^t = \frac{e^t}{K} + C$$

$$z = \frac{1}{K} + Ce^{-t}$$

$z = y^{-1}$

$$\frac{1}{y} = \frac{1}{K} + Ce^{-t}$$

is the general solution.

To find solution to IVP, we take

$t = 0$ & $y = y_0$.

Then

$$\frac{1}{y_0} = \frac{1}{K} + C \Rightarrow C = \frac{1}{y_0} - \frac{1}{K}$$

So solution of IVP is

$$\frac{1}{y} = \frac{1}{K} + \left(\frac{1}{y_0} - \frac{1}{K}\right)e^{-t}$$

$$= \frac{1}{\frac{1}{K} + \left(\frac{1}{y_0} - \frac{1}{K}\right)e^{-t}}$$

$$\Rightarrow \lim_{t\to\infty} y(t) = \frac{1}{1/K} = K.$$

3. Consider the initial value problem

$$\begin{cases} y' = \dfrac{x - \cos y}{\sin y}, \\ y(0) = \dfrac{\pi}{2}. \end{cases} \tag{2}$$

(a) (5 points) Is it linear or nonlinear? Explain.

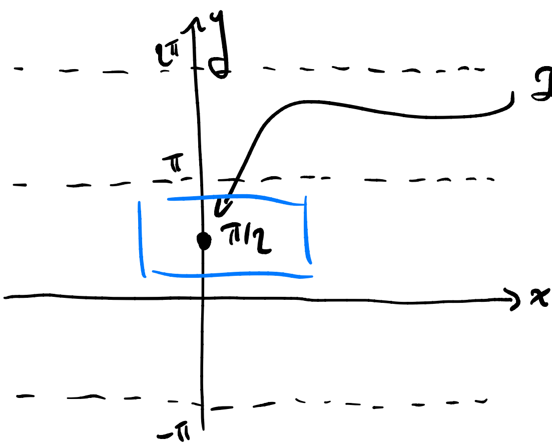
Nonlinear, because it is not in the form

$$y' + p(t)y = q(t).$$

(b) (10 points) By refering to the related existence–uniqueness theorem, what can you say about existence and uniqueness of a solution? Explain.

$f(x,y) = \dfrac{x - \cos y}{\sin y}$ and $\dfrac{\partial f}{\partial x} = -\dfrac{x\cos y}{\sin^2 y} + \dfrac{1}{\sin^2 y}$ are continuous

in $\{(x,y) \mid x \in \mathbb{R},\ y \in \mathbb{R}/\{k\pi\},\ k \in \mathbb{Z}\}$.



Initial condition belongs to $\mathbb{R} \times (0, \pi/2)$. So there exists a unique solution in a rectangular region which lies in $\mathbb{R} \times (0, \pi/2)$.

(c) (15 points) Solve the initial value problem (2).
Hint: You may need to re-express the main equation in a more convenient way.

$$\frac{dy}{dx} = \frac{x - \cos y}{\sin y} \Rightarrow dy \sin y = dx(x - \cos y)$$

$$\Rightarrow dy \underbrace{\sin y}_{N(x,y)} + dx \underbrace{(\cos y - x)}_{M(x,y)} = 0$$

$$M_y = -\sin y \neq 0 = N_x \Rightarrow \text{not exact}$$

Integration Factor:

$$\frac{M_y - N_x}{N} = \frac{-\sin y - 0}{\sin y} = \underbrace{-1}_{\text{function of } x} \Rightarrow \mu(x) = e^{\int (-1)\,dx} = e^{-x}$$

We multiply the equation by e^{-x} to get

$$\underbrace{e^{-x}\sin y}_{\tilde{N}(x,y)}\,dy + (\underbrace{e^{-x}\cos y - e^{-x}x}_{\tilde{M}(x,y)})\,dx = 0$$

$$\begin{tikzcd} & \psi(x,y) \arrow[dl, "\frac{\partial}{\partial x}"'] \arrow[dr, "\partial/\partial y"] & \\ \tilde{M}(x,y) \arrow[ur, bend right, "\text{Step I}"'] & & \tilde{N}(x,y) \arrow[ul, bend left, "\text{Step II}"] \end{tikzcd}$$

Step I $\int (e^{-x}\cos y - e^{-x}x)\,dx$

$= -e^{-x}\cos y - e^{-x}(x+1) + g(y)$

Step II

$$\frac{\partial}{\partial y}\left(-e^{-x}\cos y - e^{-x}(x+1) + g(y)\right) = e^{-x}\sin y + g'(y) = \tilde{N}$$

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C$

$\Rightarrow -e^{-x}\cos y - e^{-x}(x+1) + C = 0$

We employ the I.C. $x=0$, $y=\frac{\pi}{2}$:

$-0 - (0+1) + C = 0 \Rightarrow C = 1.$

So soln. to IVP is

$$e^{-x}\cos y + e^{-x}(x+1) = 1$$

4. Consider the differential equation

$$y'' - y' - 2y = 0, \quad t > 0. \tag{3}$$

(a) (5 points) Classify the equation.

* Constant coef.
* 2nd order
* linear

(b) (8 points) Find the fundamental set of solutions of (3). Verify your answer.

Characteristic equation: $r^2 - r - 2 = 0 \Rightarrow r_1 = -1, \quad r_2 = 2$

$\Rightarrow y_1(t) = e^{-t}, \quad y_2(t) = e^{2t}$

Verification: $W(e^{-t}, e^{2t}) = \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 2e^{t} + 2e^{t} = 4e^{t} \neq 0 \quad \forall t$ ✓

(c) (7 points) Solve the differential equation (3) together with the initial conditions $y(0) = 2$ and $y'(0) = \alpha$, where α is a constant.

General solution.

$y(t) = c_1 e^{-t} + c_2 e^{2t}$

$y'(t) = -c_1 e^{-t} + 2c_2 e^{2t}$

$\Rightarrow y(0) = 2 = c_1 + c_2$

$y'(0) = \alpha = -c_1 + 2c_2$

$\Rightarrow c_2 = \frac{\alpha + 2}{3}$

$c_1 = \frac{4 - \alpha}{3}$

Solution of IVP

$y(t) = \frac{\alpha + 2}{3} e^{2t} + \frac{4 - \alpha}{3} e^{-t}$

(d) (5 points) For what value of α, does the solution decay to zero exponentially as $t \to \infty$? Explain and give details if necessary.

If we choose $\alpha = -2$, then coefficient in front of e^{2t} becomes zero.

$\alpha = -2 \Rightarrow \frac{\alpha + 2}{3} = 0$

$\Rightarrow y(t) = \underline{\underline{2e^{-t}}}$