17.08.2023

İzmir Institute of Technology
Math 255 Differential Equations, Summer 2023
Midterm Examination II

Name: Solution Key

Student ID:

Department:

Duration: 105 Minutes

This exam contains 6 pages (check), including this page. Organize your work in the space provided. If necessary, you will be provided one empty sheet. You will also be provided Laplace transform table sheet.

- You may not use books, notes or any calculator.
- A correct answer presented without any calculation will receive no credit.
- A correct answer without any explanations will not recieve full credit. You are expected to clarify/explain your work as much as you can.
- An incorrect answer including partially correct calculations/explanations will receive partial credit.
- You are expected justify your claims unless you are using results from the lecture. Claims without any clarification will not be scored.

Grade Table

Question:	1	2	3	4	5	Total
Points:	20	25	20	20	15	100
Score:						

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1. Consider the differential equation

$$y''' + 4y' = t + \cos(2t). \tag{1}$$

(a) (10 points) Classify it. Then find its homogeneous solution $y_h(t)$.

Homogeneous part : $y''' + 4y' = 0$

Characteristic equation: $m^3 + 4m = 0 \Rightarrow m(m^2+4) = 0$

$\Rightarrow m_1 = 0, \quad m_2 = 2i, \quad m_3 = -2i$

Homogeneous solution: $y_h(t) = c_1 + c_2\cos(2t) + c_3\sin(2t)$

(b) (10 points) Use the method of undetermined coefficients to determine a suitable form for the particular solution $Y(t)$ (do not evaluate the constants).

Nonhomogeneous part: $g(t) = \underbrace{t}_{①} + \underbrace{\cos(2t)}_{②}$

① t: $Y_1(t) = At + B$ → linearly dependent with c_1

$\downarrow$ update

$Y_1(t) = At^2 + Bt$

② $\cos(2t)$: $Y_2(t) = \overbrace{C\cos 2t + D\sin 2t}$ → linearly dependent with $c_2\cos(2t) + c_3\sin(2t)$

$\downarrow$ update

$Y_2(t) = Ct\cos 2t + Dt\sin 2t$

$\Rightarrow$ Particular solution: $Y(t) = At^2 + Bt + Ct\cos 2t + Dt\sin 2t$

2. Consider the differential equation

$$x^2y'' - 2y = 4x^3 - 3, \quad x > 0. \tag{2}$$

(a) (5 points) Show that $x_0 = 0$ is a regular singular point of the equation (2) by referring to the definition.

$P(x) = x^2$, $Q(x) = 0$, $R(x) = -2$

$\lim_{x\to 0} x\,\frac{Q(x)}{P(x)} = 0$, finite

$\lim_{x\to 0} x^2\cdot\frac{-2}{x^2} = -2$, finite

$\Rightarrow$ By definition, $x_0 = 0$ is a regular singular point.

(b) (10 points) Find the homogeneous solution, $y_h(t)$, of (2).

Homogeneous part: $x^2y'' - 2y = 0$.

Let $y = x^r$. Then

$y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$.

Substitute into the homogeneous model to get

$r(r-1) - 2 = 0 \Rightarrow r^2 - r - 2 = 0$

$\Rightarrow r_1 = -1,\ r_2 = 2$

Thus

$y_h(x) = c_1x^{-1} + c_2x^2$.

(c) (10 points) Find a particular solution for (2). Combine your result with the one you found in part (b) and express the general solution.

We put the equation into the standard form to get

$$y'' - \frac{2}{x^2}y = 4x - \frac{3}{x^2}, \quad \text{so } g(x) = 4x - \frac{3}{x^2}.$$

Wronskian of the fundamental solutions is

$$W(y_1,y_2) = \begin{vmatrix} x^{-1} & x^2 \\ -x^{-2} & 2x \end{vmatrix} = 3.$$

Then, applying Variation of Parameters method we get

$$u_1(x) = -\int \frac{g(x)\, y_2(x)}{W(y_1,y_2)(x)}\,dx = -\frac{1}{3}\int \left(4x - \frac{3}{x^2}\right)x^2\,dx = -\frac{1}{3}x^4 + x,$$

$$u_2(x) = \int \frac{g(x)\, y_1(x)}{W(y_1,y_2)(x)}\,dx = \frac{1}{3}\int \left(4x - \frac{3}{x^2}\right)x^{-1}\,dx = \frac{4x}{3} + \frac{1}{2x^2},$$

which yields particular solution as

$$\begin{aligned} Y_1(x) &= y_1(x)\,u_1(x) + y_2(x)\,u_2(x) \\ &= x^{-1}\left(-\frac{1}{3}x^4 + x\right) + x^2\left(\frac{4x}{3} + \frac{1}{2x^2}\right) \\ &= \frac{3}{2} + x^3 \end{aligned}$$

So the general solution is

$$y(x) = c_1 x^{-1} + c_2 x^2 + \frac{3}{2} + x^3$$

3. Consider the differential equation

$$xy'' - y' + y = 0. \tag{3}$$

(a) (4 points) Show that $x = 0$ is a singular point of the equation (3) by referring to its definition.

$P(x) = x$ and $P(0) = 0$. So $x = 0$ is a singular point of (3).

(b) (4 points) Can you find an ordinary point of (3)? Explain.

Any x_0 with the property that $P(x_0) \neq 0$ is an ordinary point of (3). Let us choose $x_0 = 1$.

(c) (12 points) Suppose you are going to derive a series solution around the point you found in part (b). Find the associated recurrence relation.

$$y=\sum_{n=0}^{\infty} a_n (x-1)^n \Rightarrow y'=\sum_{n=1}^{\infty} n a_n (x-1)^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}.$$

Substitute:

$$x\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$x-1+1$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

all powers to $(x-1)^n$

$$\Rightarrow \sum_{n=1}^{\infty} (n+1)n\, a_{n+1} (x-1)^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

all summation index to $n=1$

$$\Rightarrow 2a_2 - a_1 + a_0 + \sum_{n=1}^{\infty} \left[n(n+1) a_{n+1} + (n+2)(n+1) a_{n+2} - (n+1) a_{n+1} + a_n \right] (x-1)^n = 0$$

$$\Rightarrow a_2 = \frac{a_1 - a_0}{2}, \quad (n+2)(n+1) a_{n+2} = a_{n+1}(n+1-n^2-n) - a_n$$

$$\Rightarrow a_{n+2} = -\frac{a_{n+1}(n^2-1) - a_n}{(n+2)(n+1)}, \quad n \geq 1.$$

4. (20 points) Use the Laplace transform to solve the initial value problem.

$$\begin{cases} y'' + 4y = e^{-t}, \\ y(0) = 1, \quad y'(0) = -1. \end{cases} \tag{4}$$

Step 1 Laplace Transform:

$$s^2 Y(s) - s\underbrace{y(0)}_{} - \underbrace{y'(0)}_{} + 4Y(s) = \frac{1}{s+1}.$$

$$\Rightarrow s^2 Y(s) - s + 1 + 4Y(s) = \frac{1}{s+1}$$

Step 2 Solve for $Y(s)$:

$$(s^2+4)Y(s) = s - 1 + \frac{1}{s+1} = \frac{s^2}{s+1}$$

$$\Rightarrow Y(s) = \frac{s^2}{(s^2+4)(s+1)} \quad \text{A}$$

Step 3. Inverse Laplace Transform:

Let us first decompose A

$$\frac{s^2}{(s^2+4)(s+1)} = \frac{As+B}{s^2+4}\,(s+1) + \frac{C}{s+1}\,(s^2+4)$$

$$\Rightarrow s^2 = As^2 + As + Bs + B + Cs^2 + 4C$$

$$\Rightarrow s^2 = (A+C)s^2 + (A+B)s + B + 4C$$

$$\Rightarrow \begin{matrix} A+C=1 \\ A+B=0 \\ B+4C=0 \end{matrix} \Rightarrow \begin{matrix} A = 4/5 \\ B = -4/5 \\ C = 1/5 \end{matrix}$$

$$\Rightarrow Y(s) = \frac{4}{5}\frac{s}{s^2+4} - \frac{4}{5}\frac{1}{s^2+4} + \frac{1}{5}\frac{1}{s+1}$$

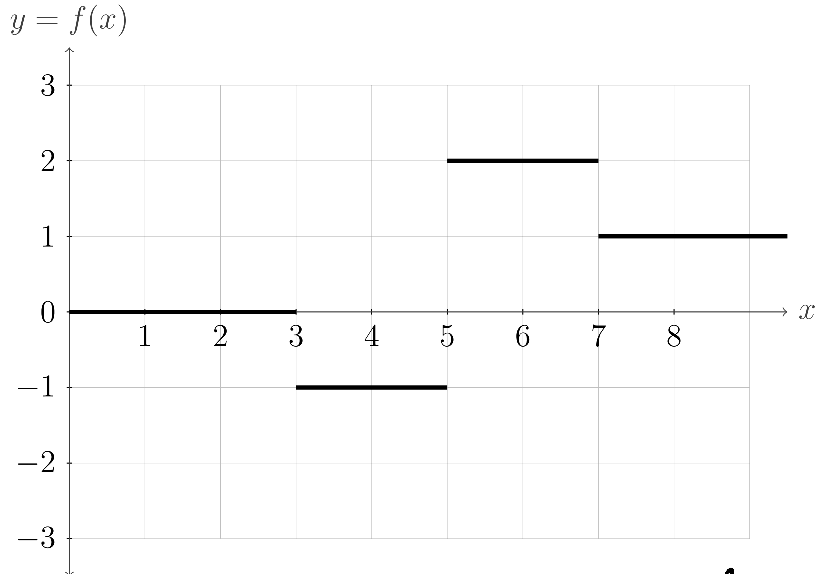
$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{4}{5}\cos(2t) - \frac{2}{5}\sin(2t) + \frac{1}{5}e^{-t}.$$

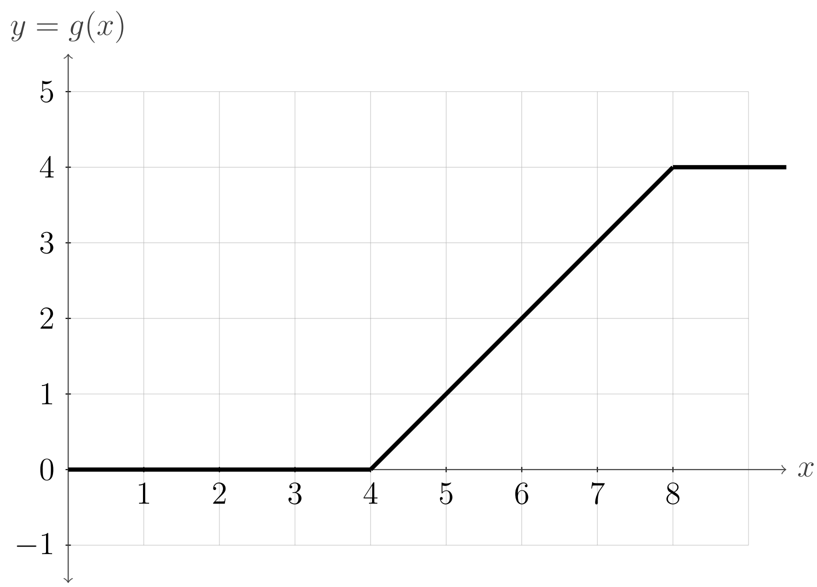
5. Express the functions whose graphs are given below in terms of unit step functions. Then, using these representations, find their Laplace transform by referring to the transform table. Give details of your work as much as you can.

(a) (8 points)



$$f(x) = -u_3(x) + 3u_5(x) - u_7(x) \Rightarrow \mathcal{L}\{f(x)\} = -\frac{e^{-3s}}{s} + 3\frac{e^{-5s}}{s} - \frac{e^{-7s}}{s}$$

(b) (7 points)



$$g(x) = u_4(x)(x-4) + u_8(x)(4-(x-4))$$

$$= \underbrace{u_4(x)}_{\substack{c=4 \\ f(x)=x}}\underbrace{(x-4)}_{f(x-4)} - \underbrace{u_8(x)}_{\substack{c=8 \\ f(x)=x}}\underbrace{(x-8)}_{f(x-8)}$$

$$\mathcal{L}\{g(x)\} = e^{-4s}\frac{1}{s^2} - e^{-8s}\frac{1}{s^2}.$$