08.01.2024

İzmir Institute of Technology
Math 255 Differential Equations, Fall 2023
Final Exam - Solution Key

Name: ____________________

Student ID: ____________________

Department: ____________________

Duration: 180 Minutes

Please read the instructions below.

- This exam contains 15 pages (check), including this page. Organize your work in the space provided.
- You may not use books, notes or any calculator.
- A correct answer presented without any calculation will receive no credit.
- A correct answer without any explanations will not recieve full credit. You are expected to clarify/explain your work as much as you can.
- An incorrect answer including partially correct calculations/explanations will receive partial credit.
- You are expected justify your claims unless you are using results from the lecture. Claims without any clarification will not be scored.

Grade Table

Question:	1	2	3	4	5	6	7	8	Total
Points:	15	20	15	15	20	10	20	10	125
Score:									

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1. (a) (5 points) (WebWork 1) Solve the differential equation

$$t^3y' + 2t^2y = 2t^5.$$

We have a first order, linear differential equation. Let us write the equation in standard form by dividing both sides by t^3:

$$y' + \frac{2}{t}y = 2t^2.$$

$p(t) = \frac{2}{t}$, so the integration factor is

$$\mu(t) = e^{\int p(t)dt} = e^{\int \left(\frac{2}{t}\right)dt} = t^2.$$

Let us multiply both sides by $\mu(t) = t^2$ and then integrate both sides:

$$\begin{aligned} y' + \frac{2}{t}y = 2t^2 \quad &\overset{\times\mu(t)}{\Longrightarrow} \quad t^2y' + 2ty = 2t^4 \\ &\Longrightarrow \quad \left(t^2y\right)' = 2t^4 \\ &\overset{\int dt}{\Longrightarrow} \quad \int \left(t^2y\right)' dt = \int 2t^4 dt \\ &\Longrightarrow \quad t^2y = \frac{2t^5}{5} + C, \quad C \in \mathbb{R} \text{ is integration constant.} \end{aligned}$$

Then, the general solution in explicit form is

$$\boxed{y(t) = \frac{2t^3}{5} + \frac{C}{t^2}}.$$

(b) (10 points) Solve the differential equation

$$x^2y'' - 2xy' + 3y = 0.$$

We seek a solution of the form $y(x) = x^r$. Differentiating y up to the order two, we get

$$y'(x) = rx^{r-1}, \quad y''(x) = r(r-1)x^{r-2}.$$

We substitute them in the equation and obtain the following quadratic equation in r:

$$r^2 - 3r - 3 = 0.$$

Roots of this equation are

$$r_1 = \frac{3}{2} + \frac{\sqrt{3}}{2}i, \quad r_2 = \frac{3}{2} - \frac{\sqrt{3}}{2}i.$$

Hence the general solution is

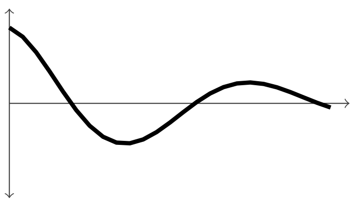
$$\boxed{y(x) = x^{\frac{3}{2}}\left(c_1 \cos\left(\ln \frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\ln \frac{\sqrt{3}}{2}x\right)\right)}.$$

2. (a) (10 points) (WebWork 2) Match the graph of the solutions shown in the figure below with each of the differential equations below.

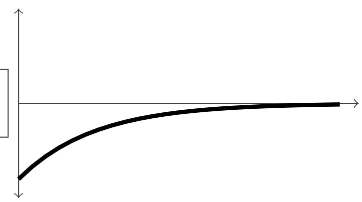
A. $y'' + 2y = 0$ **B.** $y'' - y' + 4y = 0$

C. $y'' + 5y' + 12y = 0$ **D.** $y'' + 6y' + 9y = 0$

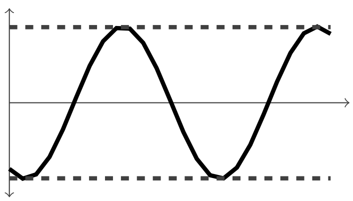
1. C



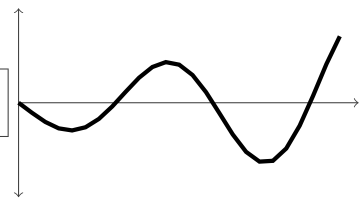
2. D



3. A



4. B



Provide details of your answers in the space below.

A.
- Characteristic equation is $r^2 + 2 = 0$.
- Roots are pure imaginary, i.e., they are complex with zero real part.
- Associated solution only involves sines and cosines, and does not involve an exponential function.

Thus solution is expected to have an oscillatory behaviour where its amplitude remains same. Answer is **3**.

B.
- Characteristic equation is $r^2 - r + 4 = 0$.
- Roots are complex with positive real parts.
- Associated solution involves sines and cosines multiplied by an exponential with a positive exponent.

Thus solution is expected to have a growing oscillatory behaviour. Answer is **4**.

C.
- Characteristic equation is $r^2 + 5r + 12 = 0$.
- Roots are complex with negative real parts.
- Associated solution involves sines and cosines multiplied by an exponential with a negative exponent.

Thus solution is expected to have a decaying oscillatory behaviour. Answer is **1**.

D.
- Characteristic equation is $r^2 + 6r + 9 = 0$.
- Roots are real and negative.
- Associated solution involves exponential, both have a negative exponent.

Thus solution is expected to have a decaying behaviour to zero. Answer is **2**.

(b) (10 points) (WebWork 2) Consider the differential equation

$$y'' + \alpha y' + \beta y = t + e^{2t}.$$

Suppose the form of the particular solution to this differential equation as prescribed by the method of undetermined coefficients is

$$y_p(t) = A_1 t^2 + A_0 t + B_0 t e^{2t}.$$

Determine the constants α and β.

As the right hand side of the differential equation is $t + e^{2t}$, an initial guess for the particular solution is supposed to be $A_1 t + A_0 + B_0 e^{2t}$. However, both $A_1 t + A_0$ and $B_0 e^{2t}$ are multiplied by t, and the particular solution is given of the form

$$y_p(t) = A_1 t^2 + A_0 t + B_0 t e^{2t}.$$

This indicates that, the parts of the initial guess A_0 and $B_0 e^{2t}$ are also solutions to the homogeneous part (i.e., they are linearly dependent with the solution to the homogeneous equation). Therefore, we infer that

$$r_1 = 0, \quad r_2 = 2$$

are roots of the associated characteristic equation. Consequently, the characteristic equation is

$$r^2 - 2r = 0,$$

and associated homogeneous part of the differential equation is

$$y'' - 2y' = 0.$$

Hence, $\boxed{\alpha = 2 \text{ and } \beta = 0}$.

3. Consider the differential equation

$$(\sin t)y'' - 2(\cos t)y' - (\sin t)y = 0, \quad 0 < t < \pi.$$

(a) (5 points) Guess a solution to this differential equation. Verify that your guess satisfies the differential equation.

An inspection on the differential equation together with the coefficients suggest that either $\sin t$ or $\cos t$ might be solutions.

In fact $y_1(t) = \cos t$ works (our motivation on this guess is the fact that with $y_1(t) = \cos t$, each term on the differential equation become same type and the equation is to be hold). Indeed differentiating y_1 up to the order two, we get

$$y_1'(t) = -\sin t, \quad y_1''(t) = -\cos t.$$

Then, we substitute y_1, y_1' and y_1'', and see that equation holds:

$$\begin{aligned} \left((\sin t)y'' - 2(\cos t)y' - (\sin t)y\right)\Big|_{y=\cos t} &= (\sin t)(-\cos t) - 2(\cos t)(-\sin t) - (\sin t)(\cos t) \\ &= 0. \end{aligned}$$

(b) (10 points) Use method of reduction of order to find the general solution.

Let us first construct a second linearly independent solution, y_2. Set $y_2(t) = v(t)\cos t$. We differentiate y_2 up to the order two and get

$$\begin{aligned} y_2(t) = v(t)\cos t \quad &\Rightarrow \quad y_2'(t) = v'(t)\cos t - v(t)\sin t \\ &\Rightarrow \quad y_2''(t) = v''(t)\cos t - 2v'(t)\sin t - v(t)\cos t. \end{aligned}$$

Now we substitute y_2, y_2' and y_2'' in the equation, and write

$$(\sin t)\left(v''\cos t - 2v'\sin t - v\cos t\right) - 2(\cos t)\left(v'\cos t - v\sin t\right) - (\sin t)(\cos t)v = 0.$$

Next, by collecting the terms, we can rewrite the above equation as

$$(\sin t)(\cos t)v'' - 2v' = 0. \quad (\bigstar)$$

Let us change the dependent variable as $v' = w$. Then $v'' = w'$ and the equation $(\bigstar)$ becomes

$$(\sin t)(\cos t)w' - 2w = 0. \quad (\bigstar\bigstar)$$

$(\bigstar\bigstar)$ is a separable differential equation and can be written as

$$\frac{1}{w}dw = \frac{2}{(\sin t)(\cos t)}dt = \frac{2\sec^2 t}{\tan t}dt.$$

We integrate both sides and get

$$\int \frac{1}{w} dw = \int \frac{2\sec^2 t}{\tan t} dt$$
$$\Rightarrow \ln|w| = 2\ln|\tan t|$$
$$\Rightarrow w = \tan^2 t.$$

Change of variables. *Let* $u = \tan t \Rightarrow du = \sec^2 t dt$. *Then*

$$\int \frac{2\sec^2 t}{\tan t} dt = 2\int \frac{1}{u} du = 2\ln|u| = 2\ln|\tan t|.$$

Then

$$\begin{aligned} v(t) &= \int w(t) dt \\ &= \int \tan^2 t dt \\ &= \int \left(\tan^2 t + 1\right) dt - \int dt \\ &= \tan t - t. \end{aligned}$$

Therefore, we find that second linearly independent solution is

$$y_2(t) = v(t)y_1(t) = (\tan t - t)\cos t = \sin t - t\cos t.$$

Hence the general solution is

$$\boxed{y_g(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 \cos t + c_2 (\sin t - t\cos t)}.$$

4. (WebWork 2) Suppose that a fourth order differential equation has a solution $y = xe^{2x}\sin x$.

   (a) (8 points) Find such a differential equation, assuming it is linear, homogeneous and has constant coefficients.

   Let us first derive the characteristic equation by obtaining the roots of the characteristic equation.

   Observe that given solution is multiple of x and $e^{2x}\sin x$. This indicates that $2+i$ is a root of the associated characteristic equation with multiplicty two. Since one root is complex, other one will be its complex conjugate, which is again with multiplicity two.

   Consequently, we deduce the following roots:

$$r_{1,2} = 2+i, \quad r_{3,4} = 2-i.$$

   Next, let us form the characteristic equation. A quadratic equation, provided sum and product of its roots, is in the following form:

$$r^2 - Sr + P = 0, \quad S = r_1 + r_2, \quad P = r_1 r_2.$$

   In our case, $S = (2+i)+(2-i) = 4$ and $P = (2+i)(2-i) = 5$. Considering the fact that each root has a multiplicity two, we obtain the characteristic equation as

$$\left(r^2 - 4r + 5\right)^2 = 0$$

   or equivalently

$$r^4 - 8r^3 + 26r^2 - 40r + 25 = 0.$$

   Consequently the differential equation associated to the above characteristic equation is as follows:

$$\boxed{y^{(4)} - 8y''' + 26y'' - 40y' + 25y = 0}.$$

   (b) (7 points) Find the general solution to this differential equation.

   Recall that roots of the characteristic equation are

$$r_{1,2} = 2+i, \quad r_{3,4} = 2-i.$$

   Then, along with the given solution $xe^{2x}\sin x$, other three linearly independent solutions are as follows:

$$y_2(x) = e^{2x}\sin x, \quad y_3(x) = e^{2x}\cos x, \quad y_4(x) = xe^{2x}\cos x.$$

   Hence the general solution is

$$\boxed{y_g(x) = c_1 xe^{2x}\sin x + c_2 e^{2x}\sin x + c_3 e^{2x}\cos x + c_4 xe^{2x}\cos x}.$$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. t^n; $\quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. t^p, $p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 6
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a\rvert$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}$, $n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)\,f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}\,f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 19
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28

5. (20 points) Find the solution of the following initial value problem

$$y'' + 4y = \sin t - u_\pi(t)\sin(t-\pi); \quad y(0) = 0, \quad y'(0) = 0.$$

Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}.$$

Applying the initial conditions we get

$$s^2Y(s) + 4Y(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}.$$

Next we solve for $Y(s)$ and obtain the transform of the solution as

$$Y(s) = \frac{1}{(s^2+4)(s^2+1)} + \frac{e^{-\pi s}}{(s^2+4)(s^2+1)}.$$

Using partial fractions,

$$\frac{1}{(s^2+4)(s^2+1)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right),$$

linearity of the inverse Laplace transform and property 5 from the table, it follows that

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{(s^2+4)(s^2+1)}\right\} &= \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\\ &= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t.\end{aligned}$$

Also, combining the properties 5 and 13 from the table, we have

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{e^{-\pi s}}{(s^2+4)(s^2+1)}\right\} &= \frac{1}{3}\left(\sin(t-\pi) - \frac{1}{2}\sin(2(t-\pi))\right)u_\pi(t)\\ &= -\left(\frac{1}{3}\sin t + \frac{1}{6}\sin 2t\right)u_\pi(t).\end{aligned}$$

Hence the solution of the IVP is

$$\boxed{y(t) = \mathscr{L}^{-1}\{Y(s)\} = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - \left(\frac{1}{3}\sin t + \frac{1}{6}\sin 2t\right)u_\pi(t)}.$$

6. (WebWork 3) Compute the inverse Laplace transform of the following functions.

(a) (5 points) $F(s) = \dfrac{3s+6}{s^2+2s-8}e^{-4s}$

Observe that the denominator is quadratic and can be factorized. So we decompose the fraction as

$$\frac{3s+6}{s^2+2s-8} = \frac{A}{s+4} + \frac{B}{s-2},$$

where A and B are to be determined. By clearing the equation of fractions and regrouping in terms of s, we get

$$\begin{aligned} 3s+6 &= A(s-2) + B(s+4) \\ &= (A+B)s - 2A + 4B \quad \Rightarrow \quad A = 1 \text{ and } B = 2. \end{aligned}$$

Then, we use linearity property of inverse Laplace transform and write

$$\mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s+4}e^{-4s}\right\} + \mathscr{L}^{-1}\left\{\frac{2}{s-2}e^{-4s}\right\}.$$

Finally, we combine the properties 1 and 13 from the table, and find that

$$\boxed{\mathscr{L}^{-1}\{F(s)\} = u_4(t)\left(e^{-4(t-4)} + 2e^{2(t-4)}\right)}.$$

(b) (5 points) $F(s) = \dfrac{s+5}{s^3-3s^2+3s-1}$

Observing that the denominator is whole cube of $(s-1)$ and, adding 1 to and subtracting 1 from numerator, we can decompose $F(s)$ as

$$F(s) = \frac{s-1+1+5}{(s-1)^3} = \frac{1}{(s-1)^2} + \frac{6}{(s-1)^3}. \tag{1}$$

Then, we use linearity of inverse Laplace transform and, combine the properties 3 and 14 from the table to derive

$$\begin{aligned} \mathscr{L}^{-1}\{F(s)\} &= \mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + \mathscr{L}^{-1}\left\{\frac{6}{(s-1)^3}\right\} \\ &= \mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + 3\mathscr{L}^{-1}\left\{\frac{2!}{(s-1)^3}\right\} \\ &= \boxed{te^t + 3t^2e^t}. \end{aligned}$$

7. (20 points) Express the general solution of the following system of equations in terms of real valued functions.

$$\dot{\mathbf{x}} = \begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix} \mathbf{x}$$

Step 1: Eigenvalues. The eigenvalue–eigenvector problem for the associated coefficients matrix is

$$\begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix} \mathbf{v} = \lambda \mathbf{v} = 0 \Rightarrow \begin{pmatrix} 2-\lambda & 5 \\ -1 & 4-\lambda \end{pmatrix} \mathbf{v} = 0. \quad (\bigstar)$$

Seeking nontrivial solutions of ($\bigstar$) (that is, eigenvectors) forces us to write

$$\det \begin{pmatrix} 2-\lambda & 5 \\ -1 & 4-\lambda \end{pmatrix} = 0,$$

which yields the characteristic equation

$$\begin{aligned} \det \left(\begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix} - \lambda \mathbf{I} \right) = 0 \;\; &\Rightarrow \;\; \det \begin{pmatrix} 2-\lambda & 5 \\ -1 & 4-\lambda \end{pmatrix} \\ &\Rightarrow \;\; (2-\lambda)(4-\lambda) + 5 = 0 \\ &\Rightarrow \;\; \lambda^2 - 6\lambda + 13 = 0. \end{aligned}$$

Then, the eigenvalues are $\lambda_1 = 3 + 2i$ and $\lambda_2 = 3 - 2i$.

Step 2: Eigenvectors. Let $\mathbf{v}^{(1)} = \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix}$ be an eigenvector for the eigenvalue λ_1. Now for $\lambda_1 = 3 + 2i$, the system ($\bigstar$) reduces to the following single equation,

$$(-1 - 2i)v_1^{(1)} + 5v_2^{(1)} = 0,$$

which has the following one parameter family of infinitely many solutions.

$$v_1^{(1)} = (1 - 2i)s, \quad v_1^{(2)} = s, \quad s \in \mathbb{R}.$$

Taking $s = 1$, we find that first eigenvector is

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 - 2i \\ 1 \end{pmatrix}.$$

Step 3: General solution. In view of above work, one (complex valued) solution is

$$e^{(3+2i)t} \begin{pmatrix} 1 - 2i \\ 1 \end{pmatrix}.$$

As a result of the following calculations

$$\begin{aligned} e^{(3+2i)t} \begin{pmatrix} 1 - 2i \\ 1 \end{pmatrix} &= e^{3t}(\cos 2t + i \sin 2t) \begin{pmatrix} 1 - 2i \\ 1 \end{pmatrix} \\ &= e^{3t} \begin{pmatrix} \cos 2t + 2\sin 2t + i(\sin 2t - 2\cos 2t) \\ \cos 2t + i \sin 2t \end{pmatrix} \\ &= e^{3t} \begin{pmatrix} \cos 2t + 2\sin 2t \\ \cos 2t \end{pmatrix} + ie^{3t} \begin{pmatrix} \sin 2t - 2\cos 2t \\ \sin 2t \end{pmatrix}, \end{aligned}$$

we separate real and imaginary parts, which together form real valued fundamental set of solutions. Hence the real valued general solution is

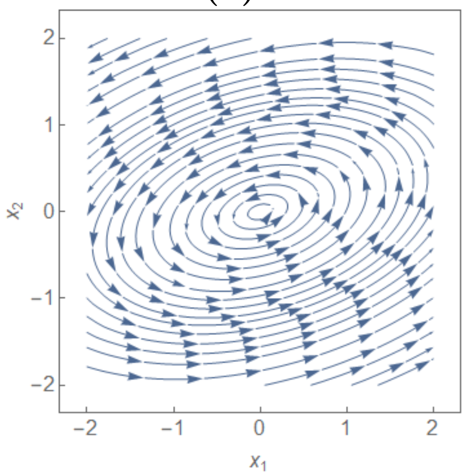
$$\boxed{\mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} \cos 2t + 2\sin 2t \\ \cos 2t \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} \sin 2t - 2\cos 2t \\ \sin 2t \end{pmatrix}}.$$

8. (10 points) (WebWork 3) Match each linear system with the one of the phase plane direction fields.

$$\mathbf{A.}\ \dot{\mathbf{x}} = \begin{pmatrix} 1 & -5 \\ 2 & -1 \end{pmatrix}\mathbf{x} \qquad \mathbf{B.}\ \dot{\mathbf{x}} = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}\mathbf{x}$$

$$\mathbf{C.}\ \dot{\mathbf{x}} = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}\mathbf{x} \qquad \mathbf{D.}\ \dot{\mathbf{x}} = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix}\mathbf{x}$$

(1)



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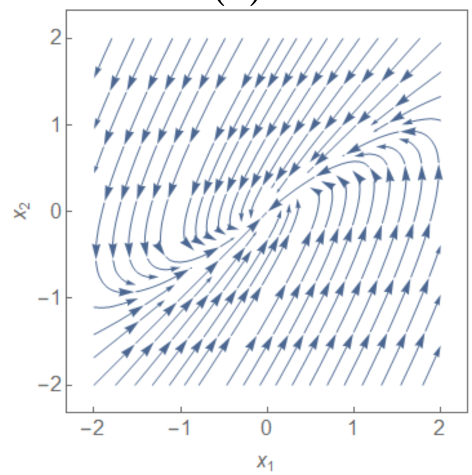

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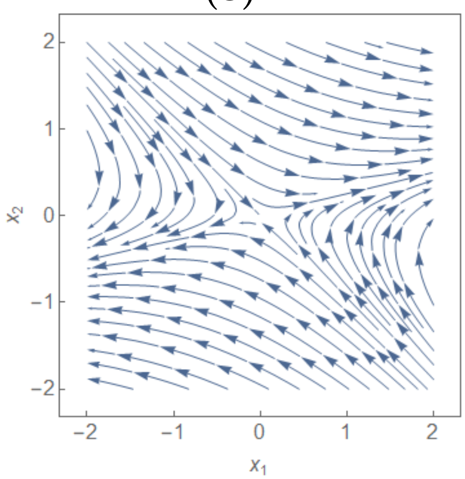

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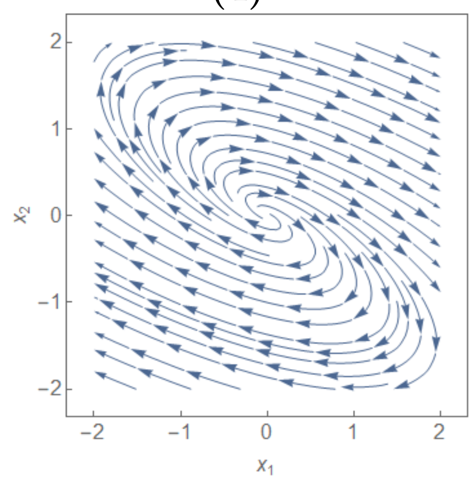

Hint: To solve this problem, you only need to compute eigenvalues In fact, it is enough to just compute whether the eigenvalues are real or complex and positive or negative.

Provide details of your answers in the space below.

A. – Characteristic equation is

$$\det\begin{pmatrix} 1-\lambda & -5 \\ 2 & -1-\lambda \end{pmatrix} = 0 \ \Rightarrow\ \lambda^2 + 9 = 0.$$

– Eigenvalues are $\lambda_1 = 3i$, $\lambda_2 = -3i$.

– Eigenvalues are pure imaginary. So the trajectories are ellipses, centered at the origin.

The linear system **A** matches with the phase plane direction field **(1)**.

B. – Characteristic equation is

$$\det\begin{pmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{pmatrix} = 0 \quad \Rightarrow \quad (\lambda+3)^2 = 0.$$

– Eigenvalues are same and $\lambda_{1,2} = -3$.

– Then the trajectories move into the origin.

The linear system **B** matches with the phase plane direction field **(2)**.

C. – Characteristic equation is

$$\det\begin{pmatrix} 1-\lambda & 4 \\ 1 & -2-\lambda \end{pmatrix} = 0 \quad \Rightarrow \quad \lambda^2 + \lambda - 6 = 0.$$

– Then eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 2$.

– Eigenvalues are of opposite sign, so the origin is a saddle point.

The linear system **C** matches with the phase plane direction field **(3)**.

D. – Characteristic equation is

$$\det\begin{pmatrix} 3-\lambda & 4 \\ -2 & -1-\lambda \end{pmatrix} = 0 \quad \Rightarrow \quad \lambda^2 + 9 = 0.$$

– Eigenvalues are $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$.

– Then the origin is spiral and trajectories emgerge from the origin.

The linear system **D** matches with the phase plane direction field **(4)**.