08.11.2023

İzmir Institute of Technology
Math 255 Differential Equations, Fall 2023
Midterm I - Solution Key

Name: ______________________________
Student ID: ______________________________
Department: ______________________________

Duration: 105 Minutes

Please read the instructions below.

- This exam contains 9 pages (check), including this page. Organize your work in the space provided.
- You may not use books, notes or any calculator.
- A correct answer presented without any calculation will receive no credit.
- A correct answer without any explanations will not recieve full credit. You are expected to clarify/explain your work as much as you can.
- An incorrect answer including partially correct calculations/explanations will receive partial credit.
- You are expected justify your claims unless you are using results from the lecture. Claims without any clarification will not be scored.

Grade Table

Question:	1	2	3	4	5	Total
Points:	20	20	15	15	30	100
Score:						

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1. (a) (10 points) (WebWork) Find the function f satisfying the differential equation

$$f'(t) - f(t) = -9t$$

and the condition $f(3) = 5$.

We have a linear, first order differential equation with unknown function f and with an initial condition $f(3) = 5$. The equation is in standard form, so $p(t) = -1$. Then the integration factor is

$$\mu(t) = e^{\int p(t)dt} = e^{\int(-1)dt} = e^{-t}.$$

Let us multiply both sides of the equation by $\mu(t) = e^{-t}$ and then integrate both sides:

$$\begin{aligned} f'(t) - f(t) = -9t \quad &\overset{\times\mu(t)}{\Longrightarrow} \quad e^{-t}f'(t) - e^{-t}f(t) = -9te^{-t} \\ &\Longrightarrow \quad \left(e^{-t}f(t)\right)' = -9te^{-t} \\ &\overset{\int dt}{\Longrightarrow} \quad \int \left(e^{-t}f(t)\right)' dt = -9\int te^{-t}dt. \end{aligned}$$

We apply integration by parts for the integral on the right hand side and get

$$\begin{aligned} e^{-t}f(t) &= -9\left(-te^{-t} + \int e^{-t}dt\right) \\ &= 9te^{-t} + 9e^{-t} + C, \end{aligned}$$

where $C \in \mathbb{R}$ is integration constant.

Integration by parts (Calculus I). *Let $u = t$ and $dv = e^{-t}dt$, then $du = dt$ and $v = -e^{-t}$. Integrating by parts, we get*

$$\begin{aligned} \int \underbrace{t}_{u}\underbrace{e^{-t}dt}_{dv} &= \underbrace{t}_{u}\underbrace{\left(-e^{-t}\right)}_{v} - \int \underbrace{\left(-e^{-t}\right)}_{v}\underbrace{dt}_{du} \\ &= -te^{-t} + \int e^{-t}dt. \end{aligned}$$

So we find that the general solution is

$$f(t) = 9(t+1) + Ce^{t}.$$

Finally we employ the initial condition $f(3) = 5$ to determine the constant C as

$$\begin{aligned} f(t) = 9(t+1) + Ce^{t} \quad &\overset{t=3}{\Longrightarrow} \quad f(3) = 9(3+1) + Ce^{3} \\ &\Longrightarrow \quad 5 = 36 + Ce^{3} \\ &\Longrightarrow \quad C = -31e^{-3}. \end{aligned}$$

Thus the solution to the initial value problem is

$$f(t) = 9(t+1) - 31e^{t-3}.$$

$\boxed{f(t) = 9(t+1) - 31e^{t-3}}$.

(b) (10 points) (WebWork) The unique solution to the initial value problem

$$y' + y = g(t), \quad y(0) = y_0$$

is $y(t) = 6e^{-2t} - 4t + 8$. Determine the constant y_0 and the function $g(t)$.

Since the function $y(t) = 6e^{-2t} - 4t + 8$ is given as the solution of the initial value problem, it must satisfy the main equation as well as the initial condition.
The initial condition is imposed at $t = 0$. So we take $t = 0$ on $y(t)$ to get

$$\begin{aligned} y(t) = 6e^{-2t} - 4t + 8 &\xRightarrow{t=0} y(0) = 6 + 8 \\ &\Longrightarrow y_0 = 14. \end{aligned}$$

$\boxed{y_0 = 14}$.

The main equation involves $y'(t)$. So let us first evaluate $y'(t)$ to find $g(t)$:

$$y(t) = 6e^{-2t} - 4t + 8 \xRightarrow{\frac{d}{dt}} y'(t) = -12e^{-2t} - 4.$$

Now we substitute $y(t)$ and $y'(t)$ into the equation and get

$$\begin{aligned} y' + y = g(t) &\Longrightarrow -12e^{-2t} - 4 + 6e^{-2t} - 4t + 8 = g(t) \\ &\Longrightarrow g(t) = -6e^{-2t} - 4t + 4. \end{aligned}$$

$\boxed{g(t) = -6e^{-2t} - 4t + 4}$.

2. (a) (10 points) (WebWork) Consider the first order differential equation

$$y' + \frac{t^2}{t^2 - 16} y = \frac{e^t}{t - 6}.$$

For each of the initial conditions below, determine the largest interval $a < t < b$ on which the existence and uniqueness theorem for the first order linear differential equations guarantees the existence of a unique solution.

We have a linear, first order differential equation. It is in standard form, and $p(t) = \frac{t^2}{t^2-16}$, $q(t) = \frac{e^t}{t-6}$. Observe that $p(t)$ is continuous on $\mathbb{R} \setminus \{-4, 4\}$ and $q(t)$ is continuous on $\mathbb{R} \setminus \{6\}$. See the table below.

<table>
<tr><td>t</td><td>$-\infty$ -4</td><td>-4 4</td><td>4 6</td><td>6 $+\infty$</td></tr>
<tr><td>Intervals on which $p(t)$ is continuous</td><td>continuous</td><td>continuous</td><td colspan="2">continuous</td></tr>
<tr><td>Intervals on which $q(t)$ is continuous</td><td colspan="3">continuous</td><td>continuous</td></tr>
<tr><td>The largest interval that contains $t_0 = 0$ on which $p(t)$ and $q(t)$ are both continuous</td><td></td><td></td><td></td><td></td></tr>
<tr><td>The largest interval that contains $t_0 = 5.5$ on which $p(t)$ and $q(t)$ are both continuous</td><td></td><td></td><td></td><td></td></tr>
</table>

- $y(0) = 0$. $\boxed{a = -4,\ b = 4}$
- $y(5.5) = -0.5$. $\boxed{a = 4,\ b = 6}$

(b) (10 points) (WebWork) Find the general solution of the differential equation

$$x^2 - 3xy + x\frac{dy}{dx} = 0.$$

We have a linear, first order differential equation with independent variable x and unknown function y. Let us write it in standard form:

$$\frac{dy}{dx} - 3y = -x.$$

Observe that $p(x) = -3$. So the integration factor is

$$\mu(x) = e^{\int p(x)dx} = e^{\int(-3)dx} = e^{-3x}.$$

$\boxed{\text{Integration factor } \mu(x) = e^{-3x}}$.

Let us multiply both sides of the equation by $\mu(x) = e^{-3x}$ and then integrate both sides:

$$\begin{aligned} \frac{dy}{dx} - 3y = -x \quad &\xrightarrow{\times\mu(t)} \quad e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = -xe^{-3x} \\ &\Longrightarrow \quad \left(e^{-3x}y\right)' = -xe^{-3x} \\ &\xrightarrow{\int dt} \quad \int \left(e^{-3x}y\right)' dx = -\int xe^{-3x}dx. \quad (\bigstar) \end{aligned}$$

We apply integration by parts for the integral on the right hand side of ($\bigstar$) and get

$$\begin{aligned} e^{-3x}y &= -\left(-\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx\right) \\ &= \frac{1}{3}xe^{-3x} + \frac{1}{9}e^{-3x} + C, \end{aligned}$$

where $C \in \mathbb{R}$ is integration constant.

Integration by parts. *Let $u = x$ and $dv = e^{-3x}dx$, then $du = dx$ and $v = -\frac{1}{3}e^{-x}$. Integrating by parts, we get*

$$\begin{aligned} \int \underbrace{x}_{u}\underbrace{e^{-3x}dt}_{dv} &= \underbrace{x}_{u}\underbrace{\left(-\frac{1}{3}e^{-3x}\right)}_{v} - \int \underbrace{\left(-\frac{1}{3}e^{-3x}\right)}_{v}\underbrace{dx}_{du} \\ &= -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx. \end{aligned}$$

Finally, multiplying both sides by e^{3x}, we obtain the general solution explicitly as

$$y = \frac{x}{3} + \frac{1}{9} + Ce^{3x}.$$

$\boxed{\text{General solution } y(x) = \frac{x}{3} + \frac{1}{9} + Ce^{3x}}$.

3. (15 points) Consider the following first-order differential equation

$$(x+3y-2)dx+(y-x-6)dy=0.$$

Firstly convert it into a homogeneous differential equation using a suitable transformation and then solve it.

Step 1. Let us rewrite the equation in the form

$$\frac{dy}{dx}=\frac{x+3y-2}{x-y+6}. \quad (\bigstar)$$

In order to convert $(\bigstar)$ into a homogeneous one, we need to get rid of the constant values appearing in the numerator and denominator. This can be done by applying the *Moving the Origin strategy.* To this end, first we find the intersection points of the lines $x+3y-2=0$ and $x-y+6=0$, i.e., we solve the following linear system of equations:

$$\begin{aligned} x+3y-2&=0,\\ x-y+6&=0. \end{aligned}$$

Solution set of this system is the point $(x,y)=(-4,2)$. Now we apply the following change of coordinates:

$$\begin{aligned} x&=X-4,\\ y&=Y+2, \end{aligned} \quad \Longrightarrow \quad \begin{aligned} dx&=dX,\\ dy&=dY. \end{aligned}$$

Then $(\bigstar)$ transforms into

$$\begin{aligned} \frac{dy}{dx}=\frac{x+3y-2}{x-y+6} \quad &\Rightarrow \quad \frac{dY}{dX}=\frac{X-4+3(Y+2)-2}{X-4-(Y+2)+6}\\ &\Rightarrow \quad \frac{dY}{dX}=\frac{X+3Y}{X-Y}. \end{aligned}$$

The function on the right hand side can be expressed as a function of Y/X. Indeed, dividing numerator and denuminator by X, we can write

$$\frac{X+3Y}{X-Y}=\frac{\frac{X+3Y}{X}}{\frac{X-Y}{X}}=\frac{1+3\left(\frac{Y}{X}\right)}{1-\left(\frac{Y}{X}\right)} \quad \Longrightarrow \quad \frac{dY}{dX}=\frac{1+3\left(\frac{Y}{X}\right)}{1-\left(\frac{Y}{X}\right)}. \quad (\bigstar\bigstar)$$

$\boxed{\text{Hence the differential equation } (\bigstar\bigstar) \text{ in } XY\text{-coordinates is homogeneous}}.$

Step 2. In order to solve the differential equation in $XY-$coordinates, we apply the following change of dependent variables:

$$Y=UX \quad \overset{\frac{d}{dX}}{\Longrightarrow} \quad Y'=U'X+U.$$

Then the differential equation $(\bigstar\bigstar)$ with unknown function $Y(X)$ transforms into the following differential equation

$$U'X+U=\frac{1+3U}{1-U}, \quad (\bigstar\bigstar\bigstar)$$

with the new unknown function $U(X)$.

Observe that (★★★) is a separable differential equation. Indeed, we can write

$$\begin{aligned} U'X + U = \frac{1+3U}{1-U} &\implies U'X = \frac{1+3U}{1-U} - U \\ &\implies U'X = \frac{1+2U+U^2}{1-U} \\ &\implies \frac{1-U}{(1+U)^2}dU + \frac{1}{X}dX = 0. \end{aligned}$$

Applying partial fraction decomposition to the first summand on the left hand side, we can write

$$\left(\frac{2}{(1+U)^2} - \frac{1}{1+U}\right)dU + \frac{1}{X}dX = 0$$

Partial fraction decomposition (Calculus I).

$$\frac{1-U}{(1+U)^2} = \frac{A}{(1+U)^2} + \frac{B}{1+U}$$
$$\Rightarrow \frac{1-U}{(1+U)^2} = \frac{A+B+BU}{(1+U)^2}.$$

This implies $A = 2, B = -1$.

Now we take antiderivative of each term with respect to U and X, respectively, and get

$$\int \left(\frac{2}{(1+U)^2} - \frac{1}{1+U}\right)dU + \int \frac{1}{X}dX = 0 \implies -\frac{2}{1+U} - \ln|1+U| + \ln|X| = C, \quad C \in \mathbb{R}.$$

This is the general solution to the equation (★★★). In order to derive the general solution of (★★), we substitute back $U = \frac{Y}{X}$ and write

$$-\frac{2}{1+\dfrac{Y}{X}} - \ln\left|1+\frac{Y}{X}\right| + \ln|X| = C.$$

Finally, to write the general solution in our original $xy-$coordinates, i.e., to derive the general solution of the (★) model, we substitute back

$$X = x+4, \quad Y = y-2,$$

and obtain

$$\boxed{-\frac{2}{1+\dfrac{y-2}{x+4}} - \ln\left|1+\frac{y-2}{x+4}\right| + \ln|x+4| = C}.$$

4. (15 points) Find the general solution of the differential equation

$$\underbrace{(y + y\cos(xy))}_{M(x,y)}\,dx + \underbrace{(x + x\cos(xy))}_{N(x,y)}\,dy = 0.$$

Denote $M(x,y) = y + y\cos(xy)$ and $N(x,y) = x + x\cos(xy)$. Let us check whether the equation is an exact differential equation or not.

$$\begin{aligned} M_y(x,y) &= 1 + \cos(xy) + y\frac{\partial}{\partial y}(\cos(xy)) \\ &= 1 + \cos(xy) + y(-\sin(xy))\frac{\partial}{\partial y}(xy) \\ &= 1 + \cos(xy) - xy\sin(xy) \end{aligned}$$

$$\begin{aligned} N_x(x,y) &= 1 + \cos(xy) + x\frac{\partial}{\partial x}(\cos(xy)) \\ &= 1 + \cos(xy) + x(-\sin(xy))\frac{\partial}{\partial x}(xy) \\ &= 1 + \cos(xy) - xy\sin(xy) \end{aligned}$$

$M_y = N_x$, and also they are continuous functions of x and y. So the equation is an exact differential equation. Now we find the general solution of the form

$$\psi(x,y) = 0,$$

where ψ is the most general function so that $\psi_x(x,y) = M(x,y)$ and $\psi_y(x,y) = N(x,y)$.

$$\begin{tikzcd} & \psi(x,y) \arrow[dl, "\frac{\partial}{\partial x}"'] \arrow[dr, "\frac{\partial}{\partial y}"] & \\ M(x,y) & & N(x,y) \end{tikzcd}$$

Step 1. To find $\psi(x,y)$, first we integrate $M(x,y)$ with respect to x:

$$\psi(x,y) = \int M(x,y)dx = \int (y + y\cos(xy))dx = xy + \sin(xy) + g(y). \quad (\bigstar)$$

Here $g(y)$ is a function of y and it is currently unknown.

Step 2. To determine $g(y)$, we differentiate $(\bigstar)$ with respect to y:

$$\frac{\partial}{\partial y}\psi(x,y) = \frac{\partial}{\partial y}\left(xy + \sin(xy) + g(y)\right) = x + x\cos(xy) + g'(y).$$

The result is equal to $N(x,y)$:

$$x + x\cos(xy) + g'(y) = N(x,y) \iff g'(y) = 0 \iff g(y) = C, \quad C \in \mathbb{R}.$$

Consequently, the general solution is given by

$$\boxed{xy + \sin(xy) + C = 0}.$$

Alternative way. Let us rewrite the differential equation in the following form:

$$y(1+\cos(xy))dx + x(1+\cos(xy))dy = 0.$$

Observe that the equation holds if $1+\cos(xy) = 0$. Else, we can cancel these terms on both summands and the equation is simplified to

$$xdy + ydx = 0.$$

Therefore, we have two cases.

Case 1. Let $1+\cos(xy) = 0$. Then

$$\boxed{xy = (2k-1)\pi, \quad k \text{ is an integer}}.$$

Case 2. Let $1+\cos(xy) \neq 0$. That is, xy is not equal to odd integer multiples of π. Then the equation is simplified to

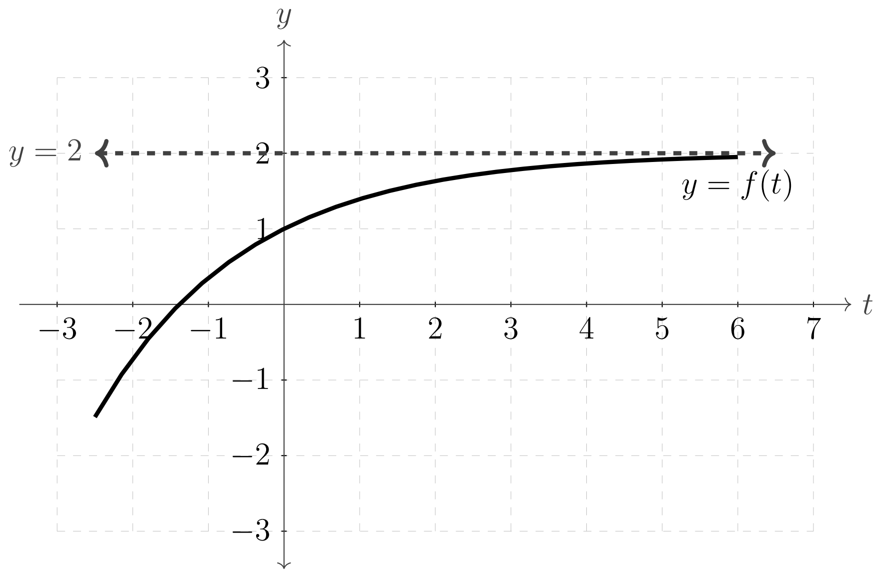
$$\begin{aligned} xdy + ydx = 0 &\implies x\frac{dy}{dx} + y = 0 \\ &\implies \frac{d}{dx}(xy) = 0. \end{aligned}$$

We integrate both sides with respect to x and get

$$\boxed{xy = C, \quad C \in \mathbb{R} \setminus \{(2k-1)\pi \,|\, k \text{ is an integer}\}}.$$

Note the general solution we found on previous page and the one we found by following the alternative way are equivalent.

5. (a) (15 points) The graph shows the solution to the initial value problem $y'(t) = -\dfrac{1}{2}y(t) + m$ with the condition $y(t_0) = 1$. $y = 2$ is the horizontal asymptote of $f(t)$.



Find t_0 and m. Give details.

According to the figure, graph of the solution passes through the point $(0, 1)$. So $t_0 = 0$. To find m, first let us classify the equation: It is first order and linear. Let us rewrite the main equation in the standard form:

$$y' + \frac{1}{2}y = m.$$

Observe that $p(t) = \frac{1}{2}$, so the integration factor is

$$\mu(t) = e^{\int p(t)dt} = e^{\int \left(\frac{1}{2}\right)dt} = e^{\frac{t}{2}}.$$

Now we multiply both sides of the equation by $\mu(t) = e^{\frac{t}{2}}$ and then integrate both sides to get

$$\begin{aligned} y' + \frac{1}{2}y = m \quad &\overset{\times\,\mu(t)}{\Longrightarrow} \quad e^{\frac{t}{2}}y' + \frac{1}{2}e^{\frac{t}{2}}y = me^{\frac{t}{2}} \\ &\Longrightarrow \quad \left(e^{\frac{t}{2}}y\right)' = me^{\frac{t}{2}} \\ &\overset{\int dt}{\Longrightarrow} \quad e^{\frac{t}{2}}y = 2me^{\frac{t}{2}} + C, \quad C \in \mathbb{R} \text{ is integration constant} \\ &\Longrightarrow \quad y = 2m + Ce^{-\frac{t}{2}}. \end{aligned}$$

According to the figure $y = 2$ is the horizontal asymptote of the solution, i.e., the limit value of the solution as $t \to \infty$. So we pass to limit on both sides as $t \to \infty$ and get

$$\lim_{t\to\infty} y = \lim_{t\to\infty} \left(2m + Ce^{-\frac{t}{2}}\right) \Longrightarrow 2 = 2m + C\underbrace{\cancel{\lim_{t\to\infty} e^{-\frac{t}{2}}}}_{\to 0} \Longrightarrow m = 1.$$

Alternative way to find m. According to the figure, solution is approaching to equilibrium solution $y = 2$, i.e., to the one with $y' = 0$. Then, from the main equation, we get

$$y' = 0 \Longrightarrow -\frac{1}{2}y + m = 0 \overset{y=2}{\Longrightarrow} m = 1.$$

$\boxed{t_0 = 0}$, $\boxed{m = 1}$.

(b) (15 points) Find the general solution of the differential equation

$$t^3 y' = y(y-t)(y+t), \quad t > 0.$$

Right hand side of the equation can be written as $y^3 - t^2 y$. So the equation takes form

$$\begin{aligned} t^3 y' = y^3 - t^2 y \implies & \; t^3 y' + t^2 y = y^3 \\ \implies & \; y' + t^{-1} y = t^{-3} y^3. \quad (\bigstar) \end{aligned}$$

($\bigstar$) is a Bernoulli type differential equation with $n = 3$. Let us apply the following change of dependent variables:

$$z = y^{1-n} = y^{-2} \overset{\frac{d}{dt}}{\implies} z' = -2y^{-3} y'.$$

Then the equation ($\bigstar$) with unknown function $y(t)$ transforms into the following linear differential equation with the new unknown $z(t)$:

$$-\frac{1}{2} z' + t^{-1} z = t^{-3} \overset{\times(-2)}{\implies} z' - 2t^{-1} z = -2t^{-3}. \quad (\bigstar\bigstar)$$

($\bigstar\bigstar$) is a first order, linear differential equation. It is in standard form and $p(t) = -2t^{-1}$. Therefore the integration factor is

$$\mu(t) = e^{\int p(t) dt} = e^{\int (-2t^{-1}) dt} = t^{-2}.$$

Now we multiply both sides of the equation by $\mu(t) = t^{-2}$ and then integrate both sides to get

$$\begin{aligned} z' - 2t^{-1} z = -2t^{-3} \overset{\times \mu(t)}{\implies} & \; t^{-2} z' - 2t^{-3} z = -2t^{-5} \\ \implies & \; \left(t^{-2} z\right)' = -2t^{-5} \\ \overset{\int dt}{\implies} & \; \int \left(t^{-2} z\right)' dt = \int \left(-2t^{-5}\right) dt \\ \implies & \; t^{-2} z = \frac{t^{-4}}{2} + C, \quad C \in \mathbb{R} \text{ is integration constant} \\ \implies & \; z = \frac{1}{2t^2} + Ct^2. \end{aligned}$$

Finally, substituting back $z = y^{-2}$ yields the general solution

$$\boxed{\frac{1}{y^2} = \frac{1}{2t^2} + Ct^2}.$$