13.12.2023

İzmir Institute of Technology
Math 255 Differential Equations, Fall 2023
Midterm II - Solution Key

Name: ______________________________

Student ID: ______________________________

Department: ______________________________

Duration: 105 Minutes

Please read the instructions below.

- This exam contains 9 pages (check), including this page. Organize your work in the space provided.
- You may not use books, notes or any calculator.
- A correct answer presented without any calculation will receive no credit.
- A correct answer without any explanations will not recieve full credit. You are expected to clarify/explain your work as much as you can.
- An incorrect answer including partially correct calculations/explanations will receive partial credit.
- You are expected justify your claims unless you are using results from the lecture. Claims without any clarification will not be scored.

Grade Table

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

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1. (a) (10 points) (WebWork) A 9th order, linear, homogeneous, constant coefficient differential equation has a characteristic equation which factors as follows.

$$(r^2 - 2r + 5)r^3(r+3)^4 = 0$$

Write the nine fundamental solutions to the differential equation as functions of the variable t.

$y_1(t) = e^t \cos 2t$ $\quad$ $y_2(t) = e^t \sin 2t$ $\quad$ $y_3(t) = 1$

$y_4(t) = t$ $\quad$ $y_5(t) = t^2$ $\quad$ $y_6(t) = e^{-2t}$

$y_7(t) = te^{-3t}$ $\quad$ $y_8(t) = t^2 e^{-3t}$ $\quad$ $y_9(t) = t^3 e^{-3t}$

Provide details of your answers in the space below.

- Roots from the first factor of the characteristic equation are $r_1 = 1 + 2i$ and $r_2 = 1 - 2i$. So associated real valued fundamental solutions are

$$\boxed{y_1(t) = e^t \cos 2t, \quad y_2(t) = e^t \sin 2t}.$$

- From the second factor, $r^3 = 0$, we have a repeated root with multiplicity 3: $r_{3,4,5} = 0$. Therefore corresponding fundamental solutions are

$$\boxed{y_3(t) = 1, \quad y_4(t) = t, \quad y_5(t) = t^2}.$$

- From the last factor, $(r+2)^4 = 0$, we have a repeated root with multiplicity 4: $r_{6,7,8,9} = -2$. Hence corresponding fundamental solutions are

$$\boxed{y_6(t) = e^{-3t}, \quad y_7(t) = te^{-3t}, \quad y_8(t) = t^2 e^{-3t} \quad y_9(t) = t^3 e^{-3t}}.$$

(b) (10 points) (WebWork) Match the following guess solutions y_p for the method of undetermined coefficients with the second-order nonhomogeneous linear equations below.

A. $y_p(x) = Ax^2 + Bx + C$ **B.** $y_p(x) = Ae^{2x}$

C. $y_p(x) = A\cos(2x) + B\sin(2x)$ **D.** $y_p(x) = (Ax + B)\cos(2x) + (Cx + D)\sin(2x)$

E. $y_p(x) = Axe^{2x}$ **F.** $y_p(x) = e^{3x}(A\cos(2x) + B\sin(2x))$

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1. **B** $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{2x}$ **2.** **A** $\dfrac{d^2y}{dx^2} + 4y = -3x^2 + 2x + 3$

3. **C** $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 13y = 3\cos(2x)$ **4.** **D** $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 15y = 3x\cos(2x)$

Provide details of your answers in the space below.

1. – Characteristic equation and roots: $r^2 - 5r + 6 = 0 \Rightarrow r_1 = 2$ and $r_2 = 3$.
   – Fundamental solutions: $y_1(x) = e^{2x}$ and $y_2(x) = e^{3x}$.

   Initial guess for a particular solution is Ae^{2x}. However, this guess is linearly dependent with $y_1(x)$. In order to ensure linearly independency, we multiply our guess by x and update it as follows:

   $$\boxed{y_p(x) = Axe^{2x}}.$$

2. – Characteristic equation: $r^2 + 4 = 0 \Rightarrow r_1 = 2i$ and $r_2 = -2i$.
   – Fundamental solutions: $y_1(x) = \cos(2x)$ and $y_2(x) = \sin(2x)$.

   Nonhomogeneous part of the equation is a polynomial of degree two. Therefore our guess for a particular solution is again of degree two and of the form

   $$\boxed{y_p(x) = Ax^2 + Bx + C}.$$

3. – Characteristic equation: $r^2 + 4r + 13 = 0 \Rightarrow r_1 = -2 + 3i$ and $r_2 = -2 - 3i$.
   – Fundamental solutions: $y_1(x) = e^{-2x}\cos(3x)$ and $y_2(x) = e^{-2x}\sin(3x)$.

   Nonhomogeneous part of the equation is $3\cos(2x)$. Thus our guess for a particular solution is of the form

   $$\boxed{y_p(x) = A\sin(2x) + B\cos(2x)}.$$

4. – Characteristic equation: $r^2 - 2r - 15 = 0 \Rightarrow r_1 = -3$ and $r_2 = 5$.
   – Fundamental solutions: $y_1(x) = e^{-3x}$ and $y_2(x) = e^{5x}$.

   We have $3x\cos(2x)$ on the right hand side of the equation. Thus our guess for a particular solution is of the form

   $$\boxed{y_p(x) = (Ax + B)\cos(2x) + (Cx + D)\sin(2x)}.$$

2. In this problem you will use variation of parameters to solve the nonhomogeneous equation

$$y'' + 5y' + 4y = 6e^{-4t}.$$

(a) (5 points) Write the characteristic equation for the associated homogeneous equation.

Homogeneous part of the differential equation is

$$y'' + 5y' + 4y = 0.$$

Then the associated characteristic equation is

$$r^2 + 5r + 4 = 0.$$

$\boxed{\text{Characteristic equation: } r^2 + 5r + 4 = 0}$.

(b) (5 points) Write the fundamental solutions for the associated homogeneous equation and their Wronskian.

We factorize left hand side of the characteristic equation and obtain the roots as

$$\begin{aligned}(r^2 + 5r + 4) = 0 &\Rightarrow (r+4)(r+1) = 0 \\ &\Rightarrow r_1 = -4, r_2 = -1.\end{aligned}$$

Then fundamental solutions for the homogeneous equation are

$$y_1(t) = e^{-4t}, \quad y_2(t) = e^{-t}.$$

Next, we differentiate y_1, y_2 and get

$$y_1'(t) = -4e^{-4t}, \quad y_2'(t) = -e^{-t}.$$

Hence the Wronskian of the fundamental solutions is

$$\begin{aligned} W(y_1, y_2)(t) &= \det\begin{pmatrix} e^{-4t} & e^{-t} \\ -4e^{-4t} & -e^{-t}\end{pmatrix} \\ &= e^{-4t} \times \left(-e^{-t}\right) - e^{-t} \times \left(-4e^{-4t}\right) \\ &= 3e^{-5t}.\end{aligned}$$

$\boxed{y_1(t) = e^{-4t}}$, $\boxed{y_2(t) = e^{-t}}$, $\boxed{W(y_1, y_2)(t) = 3e^{-5t}}$

(c) (5 points) Compute the integrals $\displaystyle\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt$ and $\displaystyle\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt$

Observe that the equation is in standard form, i.e., the coefficient of the leading order term is 1. Therefore we take $g(t) = 6e^{-4t}$. Then the first integral is

$$\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt = \int \frac{e^{-4t}\times 6e^{-4t}}{3e^{-5t}}dt = 2\int e^{-3t}dt = -\frac{2}{3}e^{-3t},$$

and the second one is

$$\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt = \int \frac{e^{-t}\times 6e^{-4t}}{3e^{-5t}}dt = 2\int dt = 2t.$$

$\boxed{\displaystyle\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt = -\frac{2}{3}e^{-3t}}$, $\boxed{\displaystyle\int \frac{y_2(t)g(t)}{W(y_1(t),y_2(t))}dt = 2t}$

(d) (5 points) Write the general solution.

From part (b), general solution to the homogeneous part is

$$y_h(t) = c_1y_1(t) + c_2y_2(t) = c_1e^{-4t} + c_2e^{-t}, \quad c_1, c_2 \in \mathbb{R}.$$

From part (c), a particular solution is

$$\begin{aligned} y_p(t) &= y_1(t)\left(-\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt\right) + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt \\ &= -2te^{-4t} - \frac{2}{3}e^{-4t}. \end{aligned}$$

Hence the general solution is

$$\begin{aligned} y_g(t) &= y_h(t) + y_p(t) \\ &= c_1e^{-4t} + c_2e^{-t} - 2te^{-4t} - \frac{2}{3}e^{-4t}. \end{aligned}$$

Combining the first and the last terms, and denoting the coefficient $(c_1 - \frac{2}{3})$ by c_1^*, we can rewrite the general solution as

$$y_g(t) = c_1^*e^{-4t} + c_2e^{-t} - 2te^{-4t}.$$

$\boxed{y_g(t) = c_1^*e^{-4t} + c_2e^{-t} - 2te^{-4t}}$

3. (20 points) Given that $y_1(t) = t$ is a solution of the differential equation

$$t^2y'' + t(t-2)y' - (t-2)y = 0, \quad t > 0,$$

solve the initial value problem

$$t^2y'' + t(t-2)y' - (t-2)y = 0; \quad y(1) = 0, \quad y'(1) = -2.$$

First let us find the second linearly independent solution. To this end, let y_2 be the second linearly independent solution and set $y_2(t) = tv(t)$. We differentiate y_2 up to the order two and get

$$\begin{aligned} y_2(t) = tv(t) \ &\Rightarrow \ y_2'(t) = v(t) + tv'(t) \\ &\Rightarrow \ y_2''(t) = 2v'(t) + tv''(t). \end{aligned}$$

Now we substitute y_2, y_2' and y_2'' in the main equation and write

$$t^2(2v' + tv'') + t(t-2)(v + tv') - (t-2)tv = 0.$$

Next, by collecting the terms, we can rewrite the above equation as

$$t^3v'' + (2t^2 + t^3 - 2t^2)v' + \cancelto{0}{(t^2 - 2t - t^2 + 2t)}v = 0 \ \Rightarrow \ v'' + v' = 0.$$

Let us change the dependent variable as $v' = w$. Then $v'' = w'$ and the $v-$equation becomes

$$w' + w = 0.$$

Solution of this $w-$equation is $w(t) = e^{-t}$. Then $v(t) = \int w(t)dt = \int e^{-t}dt = -e^{-t}$ and we find that

$$y_2(t) = tv(t) = -te^{-t}$$

is the second linearly independent solution. Thus by the superposition principle, the general solution is in the following form:

$$y_g(t) = c_1t + c_2te^{-t}, \quad c_1, c_2 \in \mathbb{R}.$$

Finally, the first initial condition requires that

$$y_g(1) = 0 \ \Rightarrow \ c_1 + c_2e^{-1} = 0.$$

To satisfy the second initial condition, we differentiate the general solution and then set $t = 1$ to get

$$\begin{aligned} y_g(t) = c_1t + c_2te^{-t} \ &\Rightarrow \ y_g'(t) = c_1 + c_2e^{-t} - c_2te^{-t} \\ &\Rightarrow \ y_g'(1) = \underbrace{c_1 + c_2e^{-1}}_{=0} - c_2e^{-1} = -2 \end{aligned}$$

This gives $c_2 = 2e$ and then $c_1 = -2$. Hence the solution to the initial value problem is

$$\boxed{y(t) = -2t + 2te^{1-t}}.$$

4. (20 points) Consider the following initial value problem:

$$y'' + 2xy' + 2y = 0; \quad y(0) = 3, \quad y'(0) = -2.$$

First, seek power series solutions of the given differential equation about the given point $x_0 = 0$; find the recurrence relation that the coefficients must satisfy. Then, find the first four nonzero terms in **each** of two linearly independent solutions $y_1(x)$ and $y_2(x)$.

We look for a solution in the form of a power series about $x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating it term by term, we obtain

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting the series for y' and y'' in the equation ($y'' + xy' + 2y = 0$) gives

$$\begin{aligned} &\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0 \\ \Rightarrow \quad &\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0. \end{aligned}$$

Next, we shift the index of summation in the first series by replacing n by n + 2 and starting the summation at 0 rather than 2,

$$\begin{aligned} &\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0 \\ \Rightarrow \quad & 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 2n a_n x^n + 2a_0 + \sum_{n=1}^{\infty} 2a_n x^n = 0 \\ \Rightarrow \quad & 2a_2 + 2a_0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 2(n+1)a_n x^n = 0. \end{aligned}$$

Again, for this equation to be satisfied for all x in some interval, the coefficients of like powers of x must be zero; hence $2a_2 + 2a_0 = 0$, and we obtain the recurrence relation $(n+2)(n+1)a_{n+2} + 2(n+1)a_n = 0$ for $n = 1, 2, 3, \ldots$, or

$$\boxed{\begin{aligned} a_2 &= -a_0 \\ a_{n+2} &= -\frac{2}{n+2}a_n \quad \text{for} \;\; n = 1, 2, 3, \ldots \end{aligned}}$$

Then we have for $n = 1, 2, 3, 4$

$$
\begin{aligned}
&a_2 = -a_0 \\
&n = 1 \to a_3 = -\frac{2}{3}a_1 \\
&n = 2 \to a_4 = -\frac{2}{4}a_2 = \frac{2}{4}a_0 \\
&n = 3 \to a_5 = -\frac{2}{5}a_3 = \frac{2^2}{5 \times 3}a_1 \\
&n = 4 \to a_6 = -\frac{2}{6}a_4 = -\frac{2^2}{6 \times 4}a_0 \\
&n = 5 \to a_7 = -\frac{2}{7}a_5 = -\frac{2^3}{7 \times 5 \times 3}a_1 \\
&\vdots
\end{aligned}
$$

The general solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \cdots \\
&= a_0 + a_1 x - a_0 x^2 - \frac{2}{3}a_1 x^3 + \frac{2}{4}a_0 x^4 + \frac{2^2}{5 \times 3}a_1 x^5 - \frac{2^2}{6 \times 4}a_0 x^6 - \frac{2^3}{7 \times 5 \times 3}a_1 x^7 + \cdots \\
&= a_0 + a_1 x - \frac{2}{2}a_0 x^2 - \frac{2}{3}a_1 x^3 + \frac{2^2}{4 \times 2}a_0 x^4 + \frac{2^2}{5 \times 3}a_1 x^5 - \frac{2^3}{6 \times 4 \times 2}a_0 x^6 - \frac{2^3}{7 \times 5 \times 3}a_1 x^7 + \cdots \\
&= a_0 \left(1 - \frac{2}{2}x^2 + \frac{2^2}{4 \times 2}x^4 - \frac{2^3}{6 \times 4 \times 2}x^6 + \cdots\right) + a_1 \left(x - \frac{2}{3}x^3 + \frac{2^2}{5 \times 3}x^5 - \frac{2^3}{7 \times 5 \times 3}x^7 + \cdots\right)
\end{aligned}
$$

Substituting initial conditions $y(0) = 3$, $y'(0) = -2$ we have $a_0 = 3$ and $a_1 = -2$. Then the final solution is $y(x) = 3y_1(x) - 2y_2(x)$ where

$$
\begin{aligned}
y_1(x) &= 1 - \frac{2}{2}x^2 + \frac{2^2}{4 \times 2}x^4 - \frac{2^3}{6 \times 4 \times 2}x^6 + \cdots \\
y_2(x) &= x - \frac{2}{3}x^3 + \frac{2^2}{5 \times 3}x^5 - \frac{2^3}{7 \times 5 \times 3}x^7 + \cdots
\end{aligned}
$$

or,

$$
\boxed{y(x) = 3\left(1 - \frac{2}{2}x^2 + \frac{2^2}{4 \times 2}x^4 - \frac{2^3}{6 \times 4 \times 2}x^6 + \cdots\right) - 2\left(x - \frac{2}{3}x^3 + \frac{2^2}{5 \times 3}x^5 - \frac{2^3}{7 \times 5 \times 3}x^7 + \cdots\right)}.
$$

5. (20 points) Find the general solution to the following Cauchy-Euler equation.

$$x^2\frac{d^2y}{dx^2} - 2y = x^2 + \ln(x), \quad x > 0.$$

We use the transformation $x := e^t$ to convert this equation into a constant coefficient differential equation. So we have

- $t = \ln(x)$
- $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \dfrac{dy}{dt}\dfrac{1}{x}$
- $\dfrac{d^2y}{dx^2} = \left[\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\right]\dfrac{1}{x^2}$

Substituting all these findings into the differential equation yields

$$x^2\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right]\frac{1}{x^2} - 2y = e^{2t} + t,$$

$$\Rightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = e^{2t} + t.$$

Now, it's a constant coefficient differential equation. The characteristic equation is $r^2 - r - 2 = 0$ and so the roots are $r_1 = -1, r_2 = 2$. Then the homogeneous solution is

$$y_h(t) = c_1e^{-t} + c_2e^{2t}, \quad c_1, c_2 \in \mathbb{R}.$$

For particular solution, we use the undetermined coefficient method (UCM). We first try $y_p(t) = Ae^{2t} + Bt + C$. But here the solution e^{2t} duplicates the first part of the homegeneous solution. We thus multiply that part by t to remove dependency. Our revised guess is then as follows.

$$y_p(t) = Ate^{2t} + Bt + C$$

We compute y_p'', y_p' and substitute into the equation. We have $A = \frac{1}{3}, B = -\frac{1}{2}, C = \frac{1}{4}$. Then the particular solution is

$$y_p(t) = \frac{te^{2t}}{3} - \frac{t}{2} + \frac{1}{4}$$

and the general solution is

$$y_g(t) = c_1e^{-t} + c_2e^{2t} + \frac{te^{2t}}{3} - \frac{t}{2} + \frac{1}{4}.$$

In terms of the original variable x, it is

$$\boxed{y_g(x) = c_1x^{-1} + c_2x^2 + \frac{x^2\ln(x)}{3} - \frac{\ln(x)}{2} + \frac{1}{4}}.$$

Alternative way As a first step, let us find the fundamental solutions to the homogeneous part of the equation

$$x^2\frac{d^2y}{dx^2} - 2y = 0.$$

We seek a solution of the form $y(x) = x^r$. Differentiating up to the order two, we get

$$y'(x) = rx^{r-1}, \quad y''(x) = r(r-1)x^{r-2}.$$

Substituting them in the equation, we obtain the following quadratic equation in r:

$$r_1 = -1, r_2 = 2.$$

Thus, fundamental solutions are $y_1(x) = x^{-1}$, $y_2(x) = x^2$ and general solution to the homogeneous part is

$$y_h(x) = c_1x^{-1} + c_2x^2.$$

As a second step, let us find a particular solution. Since we know fundamental solution, y_1 and y_2, we can apply the variation of parameters. Particular solution is given in the form

$$y_p = y_1u_1 + y_2u_2$$

where u_1 and u_2 are given by the following integrals:

$$u_1(x) = -\int \frac{y_2(x)g(x)}{W(y_1,y_2)(x)}dx, \quad u_2(x) = \int \frac{y_1(x)g(x)}{W(y_1,y_2)(x)}dx.$$

Here;

- $y_1(x) = x^{-1}$, $y_2(x) = x^2$.
- We divide both sides of the equation by x^2 to put the equation in standard form. Then we find
$$g(x) = 1 + \frac{\ln x}{x^2}.$$
- $W(y_1, y_2)$ is the Wronskian of fundamental solutions
$$W(y_1,y_2)(x) = \det\begin{pmatrix} x^{-1} & x^2 \\ -x^{-2} & 2x \end{pmatrix} = 3.$$

Then, we evaluate u_1 and u_2 as follows:

$$\begin{aligned} u_1(x) &= -\int \frac{y_2(x)g(x)}{W(y_1,y_2)(x)}dx \\ &= -\int \frac{x^2\left(1+\frac{\ln x}{x^2}\right)}{3}dx \\ &= -\frac{1}{3}\int x^2dx - \frac{1}{3}\int \ln x dx \\ &= -\frac{1}{9}x^3 - \frac{1}{3}x\ln x + \frac{1}{3}x \end{aligned}$$

Integration by parts. *Let $u = \ln x$ and $dv = dx$, then $du = \frac{1}{x}dx$ and $v = x$. Integrating by parts, we get*

$$\begin{aligned} \int \underbrace{\ln x}_{u}\underbrace{dx}_{dv} &= \underbrace{\ln x}_{u}\underbrace{x}_{v} - \int \underbrace{x}_{v}\underbrace{\frac{1}{x}}_{du}dx \\ &= x\ln x - x \end{aligned}$$

$$\begin{aligned}
u_2(x) &= \int \frac{y_1(x)g(x)}{W(y_1,y_2)(x)}dx \\
&= \int \frac{x^{-1}\left(1+\frac{\ln x}{x^2}\right)}{3}dx \\
&= \frac{1}{3}\int x^{-1}dx + \frac{1}{3}\int \frac{\ln x}{x^3}dx \\
&= \frac{1}{3}\ln x - \frac{\ln x}{6x^2} - \frac{1}{12x^2}
\end{aligned}$$

Integration by parts. *Let $u = \ln x$ and $dv = \frac{1}{x^3}dx$, then $du = \frac{1}{x}dx$ and $v = -\frac{1}{2x^2}$. Integrating by parts, we get*

$$\begin{aligned}
\int \underbrace{\ln x}_{u}\underbrace{\frac{1}{x^3}dx}_{dv} &= \underbrace{\ln x}_{u}\underbrace{\left(-\frac{1}{2x^2}\right)}_{v} - \int \underbrace{\left(-\frac{1}{2x^2}\right)}_{v}\underbrace{\frac{1}{x}}_{du}\,dx \\
&= -\frac{\ln x}{2x^2} - \frac{1}{4x^2}
\end{aligned}$$

Hence associated particular solution is

$$\begin{aligned}
y_p(x) &= y_1(x)u_1(x) + y_2(x)u_2(x) \\
&= x^{-1}\left(-\frac{1}{9}x^3 - \frac{1}{3}x\ln x + \frac{1}{3}x\right) + x^2\left(\frac{1}{3}\ln x - \frac{\ln x}{6x^2} - \frac{1}{12x^2}\right) \\
&= -\frac{x^2}{9} + \frac{1}{3}x^2\ln x - \frac{1}{2}\ln x + \frac{1}{4}
\end{aligned}$$

As a final step, we write the general solution by combining y_h and y_p:

$$\begin{aligned}
y_g(x) &= y_h(x) + y_p(x) \\
&= c_1x^{-1} + c_2x^2 - \frac{x^2}{9} + \frac{1}{3}x^2\ln x - \frac{1}{2}\ln x + \frac{1}{4}.
\end{aligned}$$

Now we combine the second and the third terms, and denoting the coefficient $(c_2 - \frac{1}{9})$ by c_2^*, we rewrite the general solution as

$$\boxed{y_g(x) = c_1x^{-1} + c_2^*x^2 + \frac{1}{3}x^2\ln x - \frac{1}{2}\ln x + \frac{1}{4}}.$$