24.10.2024

İzmir Institute of Technology
Math 255 Differential Equations, Fall 2024
Section 3 – Midterm I – Solution Key

Name: ______________________________

Student ID: ______________________________

Department: ______________________________

Duration: 105 Minutes

Please read the instructions below.

- This exam contains 8 pages (check), including this page. Organize your work in the space provided.
- You may not use books, notes or any calculator.
- A correct answer presented without any calculation will receive no credit.
- A correct answer without any explanations will not recieve full credit. You are expected to clarify/explain your work as much as you can.
- An incorrect answer including partially correct calculations/explanations will receive partial credit.
- You are expected justify your claims unless you are using results from the lecture. Claims without any clarification will not be scored.

Grade Table

Question:	1	2	3	4	5	Total
Points:	20	20	20	25	15	100
Score:						

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1. (a) (10 points) (WebWork) Consider the differential equation.

$$2x^2y + x^3y' = 1.$$

(i) Put the equation into the form $y' + p(x)y = q(x)$ and, find $p(x)$ and $q(x)$.

Multiply both sides of the equation by $\frac{1}{x^3}$ to get

$$2x^2y + x^3y' = 1 \xRightarrow{\times\frac{1}{x^3}} y' + \frac{2}{x}y = \frac{1}{x^3}.$$

$$p(x) = \frac{2}{x}, \quad q(x) = \frac{1}{x^3}$$

(ii) Find the integration factor $\mu(x)$.

$p(x) = \frac{2}{x}$, so

$$\mu(x) = e^{\int p(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2.$$

$\mu(x) = x^2$

(iii) Find the general solution $y(x)$.

Equation is first-order and linear. We multiply both sides of the standard form by $\mu(x) = x^2$, then use fundamental theorem of calculus to obtain the general solution as

$$\begin{aligned} y' + \frac{2}{x}y = \frac{1}{x^3} &\xRightarrow{\times x^2} x^2y' + 2xy = \frac{1}{x} \\ &\Longrightarrow \left(x^2y\right)' = \frac{1}{x} \\ &\xRightarrow{\int dx} \int \left(x^2y\right)' dx = \int \frac{1}{x}dx \\ &\Longrightarrow x^2y = \ln x + C, \quad C \in \mathbb{R} \\ &\Longrightarrow y(x) = \frac{\ln x + C}{x^2}. \end{aligned}$$

(b) (10 points) (WebWork) Show that the equation

$$x^2y^3 + x(1+y^2)y' = 0$$

is not exact but becomes exact when multiplied by the integration factor $\mu(x,y) = \frac{1}{xy^3}$. Solve the differential equation.

Let us write the equation in differential form as

$$x^2y^3dx + x(1+y^2)dy = 0.$$

Here, $M(x,y) = x^2y^3$ and $N(x,y) = x(1+y^2)$. Then,

$$M_y(x,y) = 3x^2y^2 \neq 1 + y^2 = N_x(x,y),$$

so the equation is not exact.

Let us multiply both sides by $\mu(x,y) = \frac{1}{xy^3}$ to obtain

$$x dx + \left(\frac{1}{y^3} + \frac{1}{y}\right) dy = 0.$$

Denote $\tilde{M}(x,y) = x$, $\tilde{N}(x,y) = \frac{1}{y^3} + \frac{1}{y}$. Then, $\tilde{M}_y(x,y) = 0 = \tilde{N}_x(x,y)$. Therefore, the equation becomes exact and, there exists a function $\psi = \psi(x,y)$ such that $\psi_x = \tilde{M}$ and $\psi_y = \tilde{N}$. To find ψ, we can follow the steps below:

Step 1: $\int \tilde{M}(x,y)dx = \int x dx = \dfrac{x^2}{2} + g(y) = \psi(x,y)$.

Step 2: $\psi_y(x,y) = \dfrac{\partial}{\partial y}\left(\dfrac{x^2}{2} + g(y)\right) = g'(y) = \tilde{N}(x,y) = \dfrac{1}{y^3} + \dfrac{1}{y}$. Then,

$$g'(y) = \frac{1}{y^3} + \frac{1}{y} \Rightarrow g(y) = -\frac{1}{2y^2} + \ln y + C, \quad C \in \mathbb{R}.$$

Hence, $\psi(x,y) = \dfrac{x^2}{2} - \dfrac{1}{2y^2} + \ln y + C$ and the general solution is given by

$$\psi(x,y) = \frac{x^2}{2} - \frac{1}{2y^2} + \ln y + C = 0.$$

2. (a) (7 points) Indicate whether the following statements are true (T) or false (F).

   (i) $y' = \dfrac{x \sin y + x}{y^2 + 1}$ is a separable differential equation. $\boxed{\text{T}}$

   (ii) $3xy + y^2 + (x^2 + 2xy)y' = 0$ is an exact equation. $\boxed{\text{F}}$

   (iii) The substitution $y = 1 + \dfrac{1}{z}$ transforms the equation

   $$y' = 2x - (3x+2)y + (x+2)y^2$$

   to a linear differential equation in z. $\boxed{\text{T}}$

   (b) (4 points) (WebWork) Which of the following are first-order linear differential equations? Use (✓) if it is, (×) otherwise.

   (i) $\dfrac{d^2y}{dx^2} + \sin(x)\dfrac{dy}{dx} = \cos x$ $\boxed{\times}$

   (ii) $x\dfrac{dy}{dx} - 4y = x^6 e^x$ $\boxed{\checkmark}$

   (iii) $\frac{dy}{dx} = y^2 - 3y$ $\boxed{\times}$

   (iv) $\left(\frac{dy}{dx}\right)^2 + y\cos x = 5$ $\boxed{\times}$

(c) (3 points) Write the order of each of the following differential equations.

(i) $(y')^4 + y''' = x^4$ $\boxed{3}$

(ii) $xyy' + 2x^2y^2 - y^3 = 0$ $\boxed{1}$

(iii) $y'' + x^2 \sin y = x^3 + 3$ $\boxed{2}$

(d) (6 points) Fig. 1 below shows the slope field for a differential equation with x as the independent variable and y as the dependent variable.

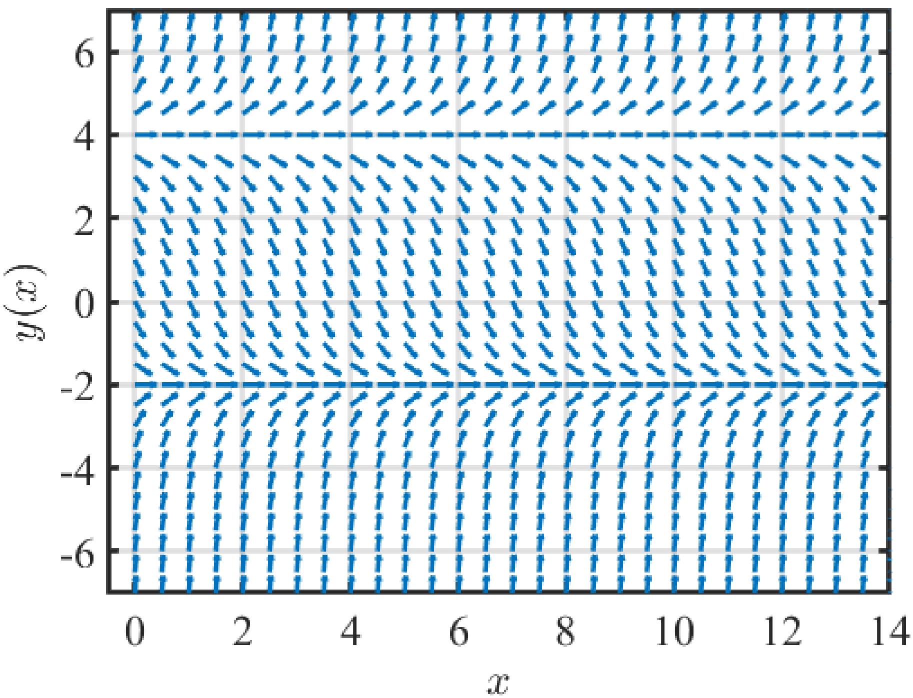

Figure 1: Slope field for a differential equation.

According to the figure, indicate whether the following statements are true (T) or false (F).

(i) The solution with initial condition $y(0) = 2$ is a decreasing function of x. $\boxed{\text{T}}$

(ii) All solutions are asymptotic to either $y = -2$ or $y = 4$. $\boxed{\text{F}}$

(iii) All solutions remain bounded. $\boxed{\text{F}}$

(iv) Among all solutions, there are two constant solutions. $\boxed{\text{T}}$

(v) Solutions with initial conditions $y(0) < -2$ are increasing functions of x. $\boxed{\text{T}}$

(iv) $y = 0$ solves the differential equation. $\boxed{\text{F}}$

3. The air in a room 20 m^3 is 3% carbon monoxide. Starting at $t = 0$, fresh air containing no carbon monoxide is blown into the room at a rate of 2 m^3/min. The air in the room flows out through a vent at the same rate.

   (a) (10 points) Form the governing initial-value problem that describes the amount of carbon monoxide as a function of time.

   Let $y(t)$ denotes the amount of carbon monoxide in the room at time t. Initially, the room contains %3 of the air. So the total amount of carbon monoxide at $t = 0$ is

$$y(0) = 20 \times \frac{3}{100} = \frac{3}{5}.$$

   The incoming air does not contain carbon monoxide, so the input rate is zero. The output rate can be found as

$$\text{output rate} = \text{concentration} \times \text{output flow} = \frac{y(t)}{20} \times 2 = \frac{y(t)}{10}.$$

   Then, using the relation

$$\text{rate of change } = \text{input rate } - \text{ output rate},$$

   together with the initial state, the initial-value problem that describes the time evolution of the amount of the carbon monoxide is given by

$$\begin{cases} y' = -\frac{y}{10}, \\ y(0) = \frac{3}{5}. \end{cases}$$

   (b) (10 points) Find the amount of carbon monoxide after 1 hour.

   General solution to the differential equation is

$$y_g(t) = Ce^{-t/10}, \quad C \in \mathbb{R}.$$

   Employing the initial condition $y(0) = \frac{3}{5}$, we get $C = \frac{3}{5}$. Hence, solution to the initial-value problem is

$$y(t) = \frac{3}{5}e^{-t/10}.$$

   To find the amount of carbon monoxide after 1 hour = 60 minutes, we take $t = 60$ and get

$$y(60) = \frac{3}{5}e^{-t/10}\Big|_{t=60} = 0.6e^{-6}.$$

4. Consider the differential equation

$$\frac{dy}{dx} = (x+y)^3 + 2(x+y) - 1. \tag{1}$$

(a) (2 points) Indicate the following:

(i) Is the equation linear (L) or nonlinear (N)? $\boxed{\text{N}}$

(ii) Write the order of the differential equation. $\boxed{1}$

(b) (5 points) Use the substitution $z = x+y$, to transform the equation (1) into a new one. Express the transformed equation.

$$z = x + y \Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx}.$$

We substitute $z = x + y$ together with $\frac{dz}{dx} = 1 + \frac{dy}{dx}$ into the equation and get

$$\frac{dz}{dx} - 1 = z^3 + 2z - 1$$

or

$$\frac{dz}{dx} - 2z = z^3.$$

(c) (3 points) According to the transformed equation, indicate the following.

(i) Transformed equation is linear $\boxed{\times}$, nonlinear $\boxed{\checkmark}$, Bernoulli $\boxed{\checkmark}$, Riccati $\boxed{\times}$ (use checkmark ($\checkmark$) if it is, crossmark ($\times$) otherwise).

(ii) Dependent variable of the transformed equation is z, independent variable of the transformed equation is x.

(d) (13 points) Solve the transformed equation.

We have a Bernoulli equation with $n = 3$. So, let us use the substitution

$$v = z^{-2} \Rightarrow \frac{dv}{dx} = -2z^{-3}\frac{dz}{dx}.$$

Then, the $z-$equation transforms into the following equation

$$\begin{aligned}\frac{dz}{dx} - 2z = z^3 &\Rightarrow \frac{1}{z^3}\frac{dz}{dx} - \frac{2}{z^2} = 1\\ &\Rightarrow -\frac{1}{2}\frac{dv}{dx} - 2v = 1\\ &\Rightarrow \frac{dv}{dx} + 4v = -2.\end{aligned}$$

Now, $v-$equation is linear with $p(x) = 4$, so the integrating factor is

$$\mu(x) = e^{\int p(x)dx}e^{\int 4dx} = e^{4x}.$$

We multiply both sides of $v-$equation by $\mu(x) = e^{4x}$ and find

$$\begin{aligned}\frac{dv}{dx} + 4v = -2 &\overset{\times e^{4x}}{\Longrightarrow} \frac{dv}{dx}e^{4x} + 4e^{4x}v = -2e^{4x}\\ &\Longrightarrow \frac{d}{dx}\left(ve^{4x}\right) = -2e^{4x}\\ &\overset{\int dx}{\Longrightarrow} \int \frac{d}{dx}\left(ve^{4x}\right)dx = -2\int e^{4x}dx\\ &\Longrightarrow ve^{4x} = -\frac{e^{4x}}{2} + C, \quad C \in \mathbb{R}\\ &\Longrightarrow v(x) = -\frac{1}{2} + Ce^{-4x}.\end{aligned}$$

Using the substitution $v = z^{-2}$ back, we find

$$\frac{1}{z^2(x)} = -\frac{1}{2} + Ce^{-4x}.$$

(e) (2 points) Use backsubstitution to find the general solution of (1).

Using the substitution $z = x + y$ back, we obtain the general solution, y, as follows

$$\frac{1}{(x + y(x))^2} = -\frac{1}{2} + Ce^{-4x}.$$

5. Consider the initial-value problem

$$\begin{cases} (x+2)\sin y + x(\cos y)y' = 0, \\ y(1) = \frac{\pi}{2}. \end{cases}$$

(a) (2 points) With a few sentences, explain the strategy you will use to derive the solution to the initial-value problem.

(b) (13 points) Solve the initial-value problem.

There are several ways to solve this initial-value problem. Below, we will give three ways to find the solution.

First way: Equation is separable. We can rewrite the equation as

$$\frac{x+2}{x} + \frac{\cos y}{\sin y} y' = 0.$$

Here $m(x) = \frac{x+2}{x}$ and $n(y) = \frac{\cos y}{\sin y}$. Their anti-derivatives, $M(x)$ and $N(y)$, in their respective variables are

$$M(x) = \int m(x)dx \int \frac{x+2}{x}dx = \int \left(1 + \frac{2}{x}\right) dx = x + 2\ln x$$

and

$$N(y) = \int n(y)dy = \int \frac{\cos y}{\sin y} dy = \ln(\sin y).$$

So the general solution is

$$M(x) + N(y) = c \Rightarrow x + 2\ln x + \ln(\sin y(x)) = c$$

It can also be rewritten as

$$\begin{aligned} x + 2\ln x + \ln(\sin y(x)) = c &\Rightarrow e^{x+2\ln x + \ln(\sin y)} = e^c \\ &\Rightarrow e^x x^2 \sin(y(x)) = C, \quad C = e^c. \end{aligned}$$

Second way: Equation can be converted to an exact equation. Let us write the equation in differential form:

$$(x+2)(\sin y)dx + x(\cos y)dy = 0.$$

Here $M(x,y) = (x+2)\sin y$, $N(x,y) = x\cos y$ and

$$M_y(x,y) = (x+2)\cos y \neq \cos y = N_x(x,y).$$

Therefore, the equation is not exact. Note that

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{(x+2)\cos y - \cos y}{x\cos y} = \frac{x+1}{x},$$

which is a function of x. So the equation can be converted to an exact one, once we multiply both sides of the equation by the integration factor

$$\mu(x) = \exp\left[\int \frac{x+1}{x}dx\right] = e^{x+\ln x} = e^x x.$$

Once we do it, the equation turns into

$$\underbrace{e^x x(x+2)\sin y}_{\tilde{M}(x,y)}\, dx + \underbrace{e^x x^2 \cos y}_{\tilde{N}(x,y)}\, dy = 0,$$

which is now an exact equation. Therefore, there exists a function $\psi = \psi(x,y)$ such that $\psi_x = \tilde{M}$ and $\psi_y = \tilde{N}$. To find ψ, we follow the steps below.

Step 1: $\int \tilde{N}(x,y)dy = \int e^x x^2(\cos y)dy = e^x x^2 \sin y + g(x) = \psi(x,y)$.

Step 2: $\psi_x(x,y) = \dfrac{\partial}{\partial x}\left(e^x x^2 \sin y + g(x)\right) = \tilde{M}(x,y)$. So, we find g as

$$\begin{aligned}\frac{\partial}{\partial x}\left(e^x x^2 \sin y + g(x)\right) = \tilde{M}(x,y) &\Rightarrow e^x(x^2+2x)\sin y + g'(x) = e^x x(x+2)\sin y\\ &\Rightarrow g'(x) = 0\\ &\Rightarrow g(x) = C, \quad C \in \mathbb{R}.\end{aligned}$$

Hence, the general solution is given by

$$\psi(x,y) = e^x x^2 \sin(y(x)) + C = 0.$$

Third way: Equation can be converted to a linear equation by a substitution. Let us change dependent variable as $z = \sin y$. Then,

$$\frac{dz}{dx} = \cos y \frac{dy}{dx}.$$

Then, we substitute z, $\frac{dz}{dx}$ and write

$$(x+2)z + xz' = 0 \Rightarrow z' + \left(1 + \frac{2}{x}\right)z = 0,$$

which is a first-order, linear equation with $p(x) = 1 + \frac{2}{x}$. Multiplying both sides by the integration factor

$$\mu(x) = \exp\left[\int \left(1 + \frac{2}{x}\right)dx\right] = e^{x + 2\ln x} = e^x x^2,$$

we get

$$\begin{aligned}z' + \left(1 + \frac{2}{x}\right)z = 0 \overset{\times \mu(x)}{\Rightarrow} e^x x^2 z' + e^x x^2\left(1 + \frac{2}{x}\right)z = 0\\ \Rightarrow \frac{d}{dx}\left(z(x)e^x x^2\right) = 0\\ \Rightarrow z(x)x^2 e^x = C, \quad C \in \mathbb{R}.\end{aligned}$$

Substituting $z = \sin y$ back, we obtain the general solution as

$$\sin(y(x))x^2 e^x = C.$$

As a conclusion, all three methods yield the same general solution. Now let us find the particular solution that satisfies $y(1) = \frac{\pi}{2}$. To this end, we take $x = 1$ and $y(x=1) = \frac{\pi}{2}$ to get

$$\sin\left(\frac{\pi}{2}\right) \times 1^2 \times e^1 = C \Rightarrow C = e.$$

Then, solution to the initial-value problem is

$$\sin(y(x))x^2 e^x = e.$$