19.11.2024

İzmir Institute of Technology
Math 255 Differential Equations, Fall 2024
Section 3 – Midterm II – Solution Key

Name: ________________________________

Student ID: ________________________________

Department: ________________________________

Duration: 105 Minutes

Please read the instructions below.

- This exam contains 7 pages (check), including this page. Organize your work in the space provided.
- You may not use books, notes or any calculator.
- A correct answer presented without any calculation will receive no credit.
- A correct answer without any explanations will not recieve full credit. You are expected to clarify/explain your work as much as you can.
- An incorrect answer including partially correct calculations/explanations will receive partial credit.
- You are expected justify your claims unless you are using results from the lecture. Claims without any clarification will not be scored.

Grade Table

Question:	1	2	3	4	5	Total
Points:	30	25	20	25	20	120
Score:						

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1. (a) (6 points) (WebWork) What values of β and A make $y = A\sin(\beta x)$ a solution to $y'' + 3y = 0$ such that $y'(0) = 4$.

The characteristic equation is $r^2 + 3 = 0$. So, the roots are pure imaginary and $r_{1,2} = \mp\sqrt{3}i$. Hence $\beta = \sqrt{3}$. To find A, we employ the condition $y'(0) = 4$:

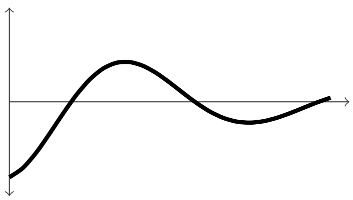
$$\begin{aligned} y(x) = A\sin(\sqrt{3}x) &\overset{\frac{d}{dx}}{\Longrightarrow} y'(x) = A\sqrt{3}\cos(\sqrt{3}x) \\ &\overset{x=0}{\Longrightarrow} y'(0) = A\sqrt{3} = 4 \\ &\Longrightarrow A = \frac{4}{\sqrt{3}}. \end{aligned}$$

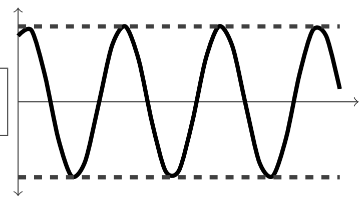
$A = \frac{4}{\sqrt{3}}, \quad \beta = \sqrt{3}$

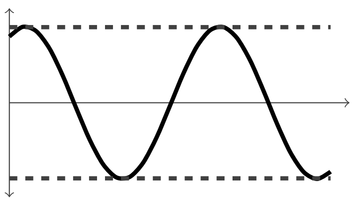
(b) (12 points) (WebWork) Match the graph of the solutions shown in the figure below with each of the differential equations below.

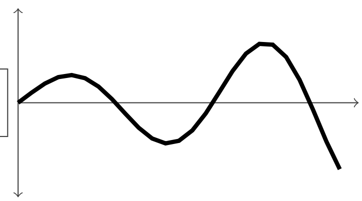
A. $2y'' + 8y = 0$ **B.** $2y'' - 2y' + 5y = 0$

C. $2y'' + 2y' + 5y = 0$ **D.** $y'' + 16y = 0$

1. $\boxed{\text{C}}$ 

2. $\boxed{\text{D}}$ 

3. $\boxed{\text{A}}$ 

4. $\boxed{\text{B}}$ 

(c) (12 points) (WebWork) Match the following guess solutions y_p for the method of undetermined coefficients with the second-order nonhomogeneous linear equations below.

A. $y_p(x) = Ax^2 + Bx + C$ **B.** $y_p(x) = Ae^{2x}$

C. $y_p(x) = A\cos(2x) + B\sin(2x)$ **D.** $y_p(x) = (Ax + B)\cos(2x) + (Cx + D)\sin(2x)$

E. $y_p(x) = Axe^{2x}$ **F.** $y_p(x) = e^{3x}(A\cos(2x) + B\sin(2x))$

1. A $\dfrac{d^2y}{dx^2} + 4y = x - \dfrac{x^2}{20}$

2. B $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 8y = e^{2x}$

3. C $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 13y = 3\cos(2x)$

4. F $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 15y = e^{3x}\cos(2x)$

2. (a) (15 points) Indicate whether the following statements are true (T) or false (F).

(i) Let $f(x) = x$ and $g(x) = x + 1$. Then f and g are linearly dependent. $\boxed{\textbf{F}}$

(ii) Let f and g be two solutions to a linear, homogeneous differential equation. Then $2f - g$ is also a solution to the equation. $\boxed{\textbf{T}}$

(iii) Suppose that the characteristic equation for a second-order, constant coefficient, linear differential equation has one positive and one negative real root. Then, there are some solutions that approach to zero asymptotically. $\boxed{\textbf{T}}$

(iv) Suppose that roots of the characteristic equation for a second-order, constant coefficient, linear differential equation are pure imaginary. Then, all solutions are bounded. $\boxed{\textbf{T}}$

(v) Let y_1 and y_2 be two solutions to the nonhomogeneous equation $y'' + 2y' + y = f(x)$ with different initial conditions. Then, their behaviour are same as $x \to \infty$. $\boxed{\textbf{T}}$

(b) (10 points) Given that $y_1(x) = xe^{-2x}$ is a solution to the constant coefficient differential equation

$$y'' + Ay' + By = 0,$$

find a second linearly independent solution y_2. Find A and B.

The equation is linear, constant coefficient and homogeneous. One solution is given in the form of an exponential function multiplied by x. This situation happens when the characteristic equation has a repeated root. Based on this observation, we infer that a second linearly independent solution is

$$y_2(x) = e^{-2x}.$$

To find A and B, we see from the exponent of y_2 that the repeated root of the characteristic equation is $r = -2$. Hence $A = 4$ and $B = 4$.

$y_2(x) = e^{-2x}, \quad A = 4, \quad B = 4$

3. (20 points) The motion of the mass $m > 0$ stretched to a spring with a spring constant $k > 0$ by taking into account the damping coefficient $b > 0$ is modeled by the initial value problem

$$\begin{cases} mu'' + bu' + ku = 0, \\ u(0) = u_0, \quad u'(0) = v_0, \end{cases}$$

where u_0 is the initial displacement of the mass and v_0 is its initial velocity.

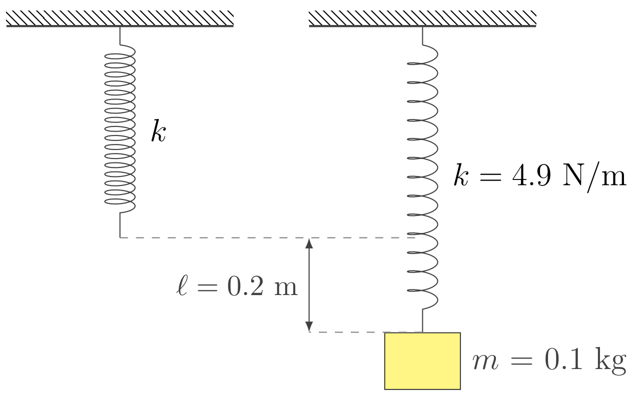
Now, suppose that a body with a mass 0.1 kg stretches a spring 0.2 m. The mass is set in motion from its equilibrium position with a downward velocity 1 m/s. Suppose that there is no damping. Formulate the differential model that describes the displacement, $u(t)$, of the mass in time t.

(Suppose that forces that cause the spring to stretch have positive sign and vice versa.)

(The standard gravity $g = 9.8$ m/s^2.)

Before the mass is set in motion, it stretches the spring by $\ell = 0.2$ m and remains at its equilibrium. In this static situation, the gravitational force and the restoring force balance each other so that

$$mg - k\ell = 0 \Rightarrow k = \frac{mg}{\ell} = \frac{0.1 \times 9.8}{0.2} = 4.9 \text{ N/m}.$$



Now, the mass is set in motion by considering the following initial states.

- The mass is released from its equilibrium position. So, the initial displacement is $u(0) = 0$.
- The mass is given an initial velocity 1 m/s with a downward direction. Notice that the downward velocity causes the spring to stretch. Therefore, with a positive sign, the second initial condition is $u'(0) = +1$ m/s.

Moreover, the problem parameters are as following.

- The mass is given by $m = 0.1$ kg.
- We consider the case where there is no damping force acting to the mass. So $b = 0$.
- Stiffness of the spring is obtained as $k = 4.9$ N/m above.

Hence, the governing initial-value problem that describes motion of the mass is

$$\begin{cases} 0.1u'' + 4.9u = 0, \\ u(0) = 0, \quad u'(0) = 1. \end{cases}$$

4. (25 points) Let y be a function that solves the differential equation

$$y'' + 3y' - 4y = e^{-4x} - 2e^{-2x}$$

and also satisfies the condition $y(0) = 1$. What should the value $y'(0)$ be so that the asymptotic behaviour $\lim_{x\to\infty} y(x) = 0$ holds true?

To determine $y'(0)$ that guarantees $\lim_{x\to\infty} y(x) = 0$, let us first find the general solution to the equation.

Step 1: Complementary solution. Associated homogeneous equation to the nonhomogeneons equation is

$$y'' + 3y' - 4y = 0.$$

Then, the characteristic equation is $r^2 + 3r - 4 = 0$ with roots $r_1 = -4$, $r_2 = 1$. Hence, the complementary part of the general solution is

$$y_c(x) = c_1 e^{-4x} + c_2 e^{x}.$$

Step 2: A particular solution. The method of undetermined coefficients leads to the following form for a particular solution:

$$y_p(x) = y_{p,1}(x) + y_{p_2}(x) = Ae^{-4x} + Be^{-2x}.$$

Notice that $y_{p,1}(x) = Ae^{-4x}$ fails to be a correct candidate for a part of particular solution because e^{-4x} is one of the fundamental solutions to the homogeneous part. To get rid of this duplication, we update $y_{p,1}(x)$ by multiplying it by x so that our correct form for a particular solution becomes

$$y_p(x) = Axe^{-4x} + Be^{-2x}.$$

Then,

$$\begin{aligned} y_p'(x) &= Ae^{-4x} - 4Axe^{-4x} - 2Be^{-2x}, \\ y_p''(x) &= -8Ae^{-4x} + 16Axe^{-4x} + 4Be^{-2x}. \end{aligned}$$

Substituting y_p, y_p' and y_p'' into the nonhomogeneous equation, we get

$$\left(-8Ae^{-4x} + 16Axe^{-4x} + 4Be^{-2x}\right) + 3\left(Ae^{-4x} - 4Axe^{-4x} - 2Be^{-2x}\right) \\ - 4\left(Axe^{-4x} + Be^{-2x}\right) = e^{-4x} - 2e^{-2x}$$

or

$$(16A - 12A - 4A)xe^{-4x} + (-8A + 3A)e^{-4x} + (4B - 6B - 4B)e^{-2x} = e^{-4x} - 2e^{-2x}.$$

We find that $A = -\frac{1}{9}$ and $B = \frac{1}{3}$, so a particular solution is

$$y_p(x) = -\frac{1}{5}xe^{-4x} + \frac{1}{3}e^{-2x}.$$

Step 3: The general solution.

$$y_g(x) = y_c(x) + y_p(x) = c_1 e^{-4x} + c_2 e^{x} - \frac{1}{5}xe^{-4x} + \frac{1}{3}e^{-2x}.$$

Next, we employ the initial condition $y(0) = 1$

$$y(0) = 1 \Rightarrow c_1 + c_2 + \frac{1}{3} = 1 \Rightarrow c_1 = \frac{2}{3} - c_2,$$

so that, solution to the initial-value problem becomes

$$y(x) = \left(\frac{2}{3} - c_2\right) e^{-4x} + c_2 e^{x} - \frac{1}{5} x e^{-4x} + \frac{1}{3} e^{-2x}.$$

Observe that each term tends to zero as $x \to \infty$ except the second one. Therefore, the prescribed value for $y'(0)$ must lead to $c_2 = 0$. Then,

$$\begin{aligned}
c_2 = 0 \Leftrightarrow y(x) &= \frac{2}{3}e^{-4x} - \frac{1}{5}xe^{-4x} + \frac{1}{3}e^{-2x} \\
\Leftrightarrow y'(x) &= -\frac{8}{3}e^{-4x} - \frac{1}{5}e^{-4x} + \frac{4}{5}xe^{-4x} - \frac{2}{3}e^{-2x} \\
\Leftrightarrow y'(0) &= -\frac{8}{3} - \frac{1}{5} - \frac{2}{3} \\
\Leftrightarrow y'(0) &= -\frac{53}{15}.
\end{aligned}$$

5. Consider the differential equation

$$y'' - 2y' + y = f(x),$$

where $f(x) = \dfrac{e^x}{1+x^2}$. Find its general solution by following the steps below:

(a) (6 points) Find two linearly independent solutions to the homogeneous equation and write the complementary solution (i.e., the general solution to the homogeneous equation).

Homogeneous part of the equation is

$$y'' - 2y' + y = 0.$$

The characteristic equation for the homogeneous equation is $r^2 - 2r + 1 = 0$ with a repeated root $r_{1,2} = 1$. Hence, linearly independent two solutions are

$$y_1(x) = e^x, \quad y_2(x) = xe^x$$

and the complementary part to the general solution is

$$y_c(x) = c_1e^x + c_2xe^x, \quad c_1, c_2 \in \mathbb{R}.$$

$y_1(x) = e^x, \quad y_2(x) = xe^x, \quad y_c(x) = c_1e^x + c_2xe^x$

(b) (2 points) Evaluate their Wronskian $W[y_1(x), y_2(x)]$.

$y_1'(x) = e^x$, $y_2'(x) = e^x + xe^x$. Then,

$$W[e^x, xe^x] = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^x(e^x + xe^x) - xe^{2x} = e^{2x}.$$

$W[y_1(x), y_2(x)] = e^{2x}$

(c) (10 points) Compute the integrals

$$\int \frac{y_1(x)f(x)}{W[y_1(x), y_2(x)]}dx, \quad \int \frac{y_2(x)f(x)}{W[y_1(x), y_2(x)]}dx$$

and use variation of parameters method to find a particular solution.

$y_1(x) = e^x$, $y_2(x) = xe^x$, $f(x) = \dfrac{e^x}{1+x^2}$ and $W[y_1(x), y_2(x)] = e^{2x}$. Then, by direct integration, the first integral can be evaluated as

$$\int \frac{y_1(x)f(x)}{W[y_1(x), y_2(x)]}dx = \int \frac{e^{2x}}{e^{2x}(1+x^2)}dx = \int \frac{1}{1+x^2}dx = \arctan x.$$

To evaluate the second integral, first we write

$$\int \frac{y_2(x)f(x)}{W[y_1(x),y_2(x)]}dx = \int \frac{xe^{2x}}{e^{2x}(1+x^2)}dx = \int \frac{x}{1+x^2}dx.$$

Now we change variables as

$$t = 1+x^2 \Rightarrow dt = 2xdx \Rightarrow \frac{1}{2}dt = xdx.$$

Then, by transforming the integral in the new variable t, we find

$$\int \frac{y_2(x)f(x)}{W[y_1(x),y_2(x)]}dx = \int \frac{x}{1+x^2}dx = \frac{1}{2}\int \frac{1}{t}dt = \frac{1}{2}\ln|t| = \frac{\ln(1+x^2)}{2}.$$

Now the method of variation of parameters leads to a particular solution y_p in the following form

$$\begin{aligned} y_p(x) &= y_1(x)u_1(x) + y_2(x)u_2(x) \\ &= e^x\left(-\int \frac{y_2(x)f(x)}{W[y_1(x),y_2(x)]}dx\right) + xe^x\left(\int \frac{y_1(x)f(x)}{W[y_1(x),y_2(x)]}dx\right) \\ &= -\frac{e^x\ln(1+x^2)}{2} + xe^x\arctan x \end{aligned}$$

$$\int \frac{y_1(x)f(x)}{W[y_1(x),y_2(x)]}dx = \arctan x, \qquad \int \frac{y_2(x)f(x)}{W[y_1(x),y_2(x)]}dx = \frac{\ln(1+x^2)}{2},$$

$$y_p(x) = -\frac{e^x\ln(1+x^2)}{2} + xe^x\arctan x$$

(d) (2 points) Write the general solution.

$$y_g(x) = c_1e^x + c_2xe^x - \frac{e^x\ln(1+x^2)}{2} + xe^x\arctan x$$