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İzmir Institute of Technology
Math 255 Differential Equations, Fall 2024
Section 3 – Midterm III – Solution Key

Name: ______________________________

Student ID: ______________________________

Department: ______________________________

Duration: 105 Minutes

Please read the instructions below.

- This exam contains 9 pages (check), including this page. Organize your work in the space provided.
- You may not use books, notes or any calculator.
- A correct answer presented without any calculation will receive no credit.
- A correct answer without any explanations will not recieve full credit. You are expected to clarify/explain your work as much as you can.
- An incorrect answer including partially correct calculations/explanations will receive partial credit.
- You are expected justify your claims unless you are using results from the lecture. Claims without any clarification will not be scored.

Grade Table

Question:	1	2	3	4	5	Total
Points:	30	10	20	15	25	100
Score:						

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1. (a) (10 points) (WebWork) Suppose that the characteristic equation for a differential equation is $(r^2+4)^2(r+1)^2 = 0$.

   (i) Find such a differential equation, assuming it is homogeneous and has constant coefficients.

$$(r^2+4)^2(r+1)^2 = r^6 + 2r^5 + 9r^4 + 16r^3 + 24r^2 + 32r + 16,$$

hence the associated differential equation is

$$y^{(6)} + 2y^{(5)} + 9y^{(4)} + 16y''' + 24y'' + 32y' + 16y = 0$$

(ii) Find the general solution to this differential equation.
From the factorized form of the characteristic equation, we find the roots as

$$r_{1,2} = 2i, \quad r_{3,4} = -2i, \quad r_{5,6} = -1.$$

Therefore, six linearly independent solutions are

$$\begin{aligned} y_1(x) &= \cos(2x), \\ y_2(x) &= \sin(2x), \\ y_3(x) &= x\cos(2x), \\ y_4(x) &= x\sin(2x), \\ y_5(x) &= e^{-x}, \\ y_6(x) &= xe^{-x}. \end{aligned}$$

Hence, the general solution is

$$y(x) = c_1\cos(2x) + c_2\sin(2x) + c_3x\cos(2x) + c_4x\sin(2x) + c_5e^{-x} + c_6xe^{-x}.$$

(b) (5 points) Consider the nonhomogeneous differential equation with the same homogeneous part as described in Part (a) together with a nonhomogeneous term

$$f(x) = \sin(x) + \sin(2x).$$

Find an appropriate form for a particular solution, if the method of undetermined coefficients is to be applied (do not evaluate the coefficients).
Splitting the nonhomogeneous part of the differential equation by $\sin(x)$ and $\sin(2x)$, our guess is of the form

$$y_p(x) = y_{p,1}(x) + y_{p,2}(x),$$

where

$$y_{p,1}(x) = A\cos(x) + B\sin(x), \quad y_{p,2}(x) = C\cos(2x) + D\sin(2x).$$

However, the sumands in $y_{p,2}$ are linearly independent with y_1 and y_2, respectively. In order get rid of this duplication, we modify $y_{p,2}$ by multiplying it by x^2. Hence, a true form for a particular solution is

$$y_p(x) = A\cos(x) + B\sin(x) + Cx^2\cos(2x) + Dx^2\sin(2x).$$

(c) (10 points) (WebWork) Consider the second-order differential equation

$$u'' + 4u' + 8u = 2\sin(2x).$$

Without solving it, rewrite the differential equation as an equivalent system of first-order equations.

Let $y_1 = u$ and $y_2 = u'$. Then, it follows that

$$y_1' = y_2.$$

Differentiating $y_2 = u'$, we get $y_2' = u''$. We use the given equation and write

$$\begin{aligned} y_2' = u'' &= -4u' - 8u + 2\sin(2x) \\ &= -4y_2 - 8y_1 + 2\sin(2x). \end{aligned}$$

Consequently, we obtain the following system of first-order equations

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -8y_1 - 4y_2 + 2\sin(2x), \end{aligned}$$

(d) (5 points) Indicate whether the following statements are true (T) or false (F).

(i) $x_0 = 0$ is an ordinary point for the differential equation $xy'' + (2x-1)y' + 3y = 0$. $\boxed{\text{F}}$

(ii) The vector functions

$$\mathbf{y}_1(x) = \begin{bmatrix} e^{-4x} \\ 2e^{-4x} \end{bmatrix}, \quad \mathbf{y}_2(x) = \begin{bmatrix} 3e^{3x} \\ -e^{3x} \end{bmatrix}$$

are linearly independent. $\boxed{\text{T}}$

(iii) Neither of the roots of $r^6 + r^2 = 0$ are pure imaginary. $\boxed{\text{T}}$

(iv) Let $\mathbf{v}$ be an eigenvector of the matrix A. Then, the vectors $\mathbf{v}$ and A$\mathbf{v}$ are linearly independent. $\boxed{\text{F}}$

(iv) Let λ be an eigenvalue of the matrix A. Then, the matrix $\text{A} - \lambda\text{I}$ is singular, where I is the identity matrix. $\boxed{\text{T}}$

2. (10 points) Given that $y_1(x) = e^x$ is a solution of the differential equation

$$xy'' - (x+1)y' + y = 0, \quad x > 0,$$

find the general solution.

First let us find the second linearly independent solution. To this end, let y_2 be the second linearly independent solution and set $y_2(x) = e^x v(x)$. We differentiate y_2 up to the order two and get

$$\begin{aligned} y_2(x) = e^x v(x) \quad &\Rightarrow \quad y_2'(x) = e^x \left(v(x) + v'(x)\right) \\ &\Rightarrow \quad y_2''(t) = e^x \left(v(x) + 2v'(x) + v''(x)\right). \end{aligned}$$

Now we substitute y_2, y_2' and y_2'' into the equation and write

$$xe^x \left(v + 2v' + v''\right) - (x+1)e^x \left(v + v'\right) + e^x v = 0.$$

Collecting the terms, we rewrite the above equation as

$$xv'' + (x-1)v' = 0.$$

Set $v' = w$. Then $v'' = w'$ and the $v-$equation becomes

$$xw' + (x-1)w = 0 \Rightarrow w' + \left(1 - \frac{1}{x}\right) w = 0.$$

Now, $w-$equation is first-order and linear with integration factor

$$\mu(x) = e^{\int \left(1 - \frac{1}{x}\right) dx} = e^x x^{-1}.$$

A solution to the $w-$equation can be obtained by using the method of integration factors and Fundamental Theorem of Calculus as follows:

$$\begin{aligned} w' + \left(1 - \frac{1}{x}\right) w = 0 \overset{\times \mu(x)}{\Longrightarrow} & \frac{d}{dx}\left(w(x)e^x x^{-1}\right) = 0 \\ \overset{\int dx}{\Longrightarrow} & w(x)e^x x^{-1} = 1 \\ \Longrightarrow & w(x) = xe^{-x}. \end{aligned}$$

Then, using integration by parts, a solution to the $v-$equation is

$$\begin{aligned} v(x) &= \int w(x)dx = \int xe^{-x} dx \\ &= -xe^{-x} - \int (-e^{-x})dx \\ &= -xe^{-x} - e^{-x}. \end{aligned}$$

Therefore, a second linearly independent solution is

$$y_2(x) = e^x v(x) = e^x \left(-xe^{-x} - e^{-x}\right) = -x - 1.$$

Hence, the general solution is

$$y_g(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 e^x + c_2(-x-1) = c_1 e^x + c_2^*(x+1), \quad c_2^* = -c_2.$$

3. (20 points) Solve the differential equation

$$(1-x)y'' + xy' - y = 0,$$

by means of a power series about the point $x = 0$ (find the first three terms in each of the linearly independent solutions).

We look for a solution in the form of a power series about $x_0 = 0$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Differentiating it term by term, we obtain

$$y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.$$

Substituting the series for y' and y'' in the equation gives

$$(1-x)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0.$$

We shift the index of the first summation by replacing n by $n+2$ and index of the second summation by replacing n by $n+1$:

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty}(n+1)nc_{n+1}x^n + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0.$$

In order to combine all series, we start the first and the fourth series from $n = 1$ (rather then $n = 0$) and write

$$2c_2 + \sum_{n=1}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty}(n+1)nc_{n+1}x^n + \sum_{n=1}^{\infty} nc_n x^n - c_0 - \sum_{n=1}^{\infty} c_n x^n = 0$$

or

$$2c_2 - c_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)c_{n+2} - n(n+1)c_{n+1} + (n-1)c_n\right]x^n = 0.$$

In order for the equation to be satisfied, we must have

$$c_2 = \frac{c_0}{2}, \quad (n+2)(n+1)c_{n+2} - n(n+1)c_{n+1} + (n-1)c_n = 0,$$

which yields the recurrence relation

$$c_{n+2} = \frac{n(n+1)c_{n+1} - (n-1)c_n}{(n+2)(n+1)}, \quad n = 1, 2, \ldots$$

From the recurrence relation, we get

$$\begin{aligned}
n = 1 \Rightarrow & c_3 = \frac{2c_2}{6} = \frac{c_0}{6}, \\
n = 2 \Rightarrow & c_4 = \frac{6c_3 - c_2}{12} = \frac{c_3}{2} - \frac{c_2}{12} = \frac{c_0}{24} \\
n = 3 \Rightarrow & c_5 = \frac{12c_4 - 2c_3}{20} = \frac{3c_4}{5} - \frac{c_3}{10} = \frac{c_0}{120} \\
& \vdots \\
& c_n = \frac{c_0}{n!}.
\end{aligned}$$

Then, the general solution is

$$\begin{aligned}
y_g(x) &= c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots \\
&= c_0 + c_1 x + \frac{c_0}{2}x^2 + \frac{c_0}{6}x^3 + \frac{c_0}{24}x^4 + \frac{c_0}{120}x^5 + \cdots + \frac{c_0}{n!}x^n + \cdots \\
&= c_0\left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!} + \cdots\right) + c_1 x.
\end{aligned}$$

Observe that the infinite sum can be rewritten as

$$\begin{aligned}
1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!} + \cdots &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!} + \cdots\right) - x \\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!} - x
\end{aligned}$$

In the above result, the series is the Taylor series expansion of e^x. Hence, the general solution takes form

$$y_g(x) = c_0(e^x - x) + c_1 x = c_0 e^x + c_1^* x, \quad c_1^* = c_1 - c_0.$$

4. Find the general solution to the differential equation

$$x^2 y'' - 3xy' + \alpha y = 0,$$

for $\alpha = 3$, $\alpha = 4$ and $\alpha = 5$.

We seek a solution of the form $y(x) = x^r$. Then, $y'(x) = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting y, y' and y'' into the equation yields the following characteristic equation:

$$r(r-1) - 3r + \alpha r = 0 \Rightarrow r^2 - 4r + \alpha = 0.$$

(a) (5 points) $\alpha = 3$:

For $\alpha = 3$, the characteristic equation becomes $r^2 - 4r + 3 = 0$, with roots $r_1 = 1$, $r_2 = 3$. Thus, the general solution is

$$y_g(x) = c_1 x + c_2 x^3.$$

(b) (5 points) $\alpha = 4$:

For $\alpha = 4$, the characteristic equation becomes $r^2 - 4r + 4 = 0$, with a repeated root $r_{1,2} = 2$. Thus, the general solution is

$$y_g(x) = c_1 x^2 + c_2 \ln x x^2.$$

(c) (5 points) $\alpha = 5$:

For $\alpha = 5$, the characteristic equation becomes $r^2 - 4r + 5 = 0$, with complex conjugate roots $r_{1,2} = 2 \mp i$. Thus, the general solution is

$$y_g(x) = x^2 \left(c_1 \cos(\ln x) + c_2 \sin(\ln x) \right).$$

5. (a) (15 points) Find the general solution to the following system of equations

$$\begin{cases} y_1' = 3y_1 - 2y_2, \\ y_2' = 2y_1 - 2y_2. \end{cases}$$

The system can be written in matrix for as $\dot{\mathbf{y}} = \mathrm{A}\mathbf{y}$, where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathrm{A} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

and $\dot{\mathbf{y}}$ denotes the derivative of the vector function $\mathbf{y}$.

Step 1: Eigenvalues. The eigenvalue-eigenvector equation for the coefficient matrix A is $(\mathrm{A} - \lambda\mathrm{I})\mathbf{v} = \mathbf{0}$. Seeking nontrivial solutions of this system leads us

$$\det(\mathrm{A} - \lambda\mathrm{I}) = \det\begin{bmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - \lambda - 2 = 0.$$

Hence, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$.

Step 2: Eigenvectors.

- $\lambda_1 = -1$: Let $\mathbf{v}^{(1)} = \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \end{bmatrix}$ be an eigenvector for the eigenvalue $\lambda_1 = -1$. Taking $\lambda_1 = -1$, the system $(\mathrm{A} - \lambda_1\mathrm{I})\mathbf{v}^{(1)} = \mathbf{0}$ reduces to the following single equation:

$$2v_1^{(1)} - v_2^{(1)} = 0,$$

which yields one parameter family of infinitely many solutions:

$$v_1^{(1)} = s, \quad v_2^{(2)} = 2s, \quad s \in \mathbb{R}.$$

Taking $s = 1$, we find that the eigenvector for the eigenvalue λ_1 as $\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

- $\lambda_1 = 2$: Let $\mathbf{v}^{(2)}$ be an eigenvector for the eigenvalue $\lambda_2 = 2$. Similar calculations leads to $\mathbf{v}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Step 3: General solution. Any superposition of $\mathbf{v}^{(1)}e^{\lambda_1 x}$ and $\mathbf{v}^{(2)}e^{\lambda_2 x}$ gives the general solution:

$$\mathbf{y}_g(x) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-x} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2x} = \begin{bmatrix} c_1e^{-x} + 2c_2e^{2x} \\ 2c_1e^{-x} + c_2e^{2x} \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

Hence,

$$y_1(x) = c_1e^{-x} + 2c_2e^{2x},$$
$$y_2(x) = 2c_1e^{-x} + c_2e^{2x}.$$

(b) (10 points) Consider a system of first-order differential equations in matrix form

$$\dot{\mathbf{y}} = \mathrm{A}\mathbf{y}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Here, A is a 2−dimensional square matrix with eigenvalues $\lambda_{1,2}$ and associated eigenvectors $\mathbf{v}_{1,2}$ as follows:

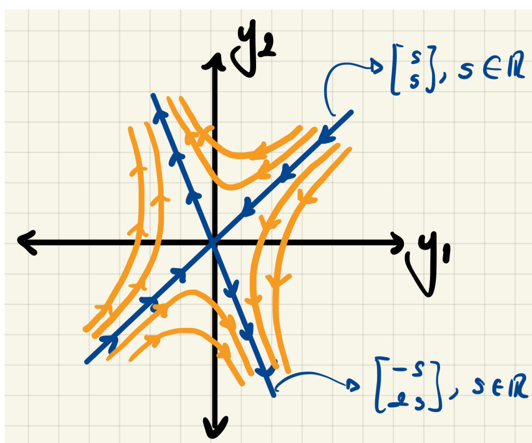
$$\lambda_1 = 2, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}; \quad \lambda_2 = -3, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(i) Write the general solution.

$$\mathbf{y}(x) = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{2x} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3x}$$

(ii) Sketch a phase portrait of the system.

- ☐ $\lambda_1 = 2 > 0$. Therefore, direction of all trajectories that are not passing through the line $y_1 - y_2 = 0$ (i.e., all solutions with $c_1 \neq 0$) are determined by the eigenvector $\mathbf{v}_1$ as $x \to \infty$.
- ☐ $\lambda_2 = -3 < 0$. Similarly, direction of all trajectories that are not passing through $2y_1 + y_2 = 0$ (i.e., all solutions with $c_2 \neq 0$) are determined by the eigenvector $\mathbf{v}_2$ as $x \to -\infty$.



(iii) According to the phase portrait, can you find an initial condition that guarantees

$$\mathbf{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

as $x \to \infty$? Explain.

From the phase-portrait, all solutions with initial states $(y_1(x_0), y_2(x_0))$ that lie on the trajectory $y_1 - y_2 = 0$ approach to the origin of the phase plane as $x \to \infty$. Based on this observation, we infer that any initial-value problem with initial conditions that are equal, i.e.

$$\dot{\mathbf{y}} = \mathrm{A}\mathbf{y}, \quad \mathbf{y}(x_0) = \begin{bmatrix} y_1(x_0) \\ y_2(x_0) \end{bmatrix}, \quad \text{with the property } y_1(x_0) = y_2(x_0)$$

assumes a solution that approaches to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $x \to \infty$.