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İzmir Institute of Technology
MSE 222 Applied Mathematics for Materials Science and Engineering, Spring 2025
Midterm I – Solution Key

Name: ______________________________

Student ID: ______________________________

Duration: 105 Minutes

Grade Table

Question:	1	2	3	4	5	Total
Points:	20	28	22	25	25	120
Score:						

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1. (a) (8 points) (WebWork) Consider the subset of $\mathbb{R}^2$ consisting of vectors $\begin{bmatrix} a \\ b \end{bmatrix}$, where a and b are **integers**. Determine whether this set is a subspace of $V = \mathbb{R}^2$ or not, by indicating whether the following statements are true or false.

   (i) The set contains the zero vector. [$\boxed{\text{True}}$ / ~~False~~]

   (ii) This set is closed under vector addition. [$\boxed{\text{True}}$ / ~~False~~]

   (iii) This set is closed under scalar multiplication. [~~True~~ / $\boxed{\text{False}}$]

   (iv) This set is a subspace. [~~True~~ / $\boxed{\text{False}}$]

   (b) (5 points) (WebWork) Let

$$\mathbf{u} = \begin{bmatrix} -4 \\ 5 \\ 5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix}.$$

   Give an example of a vector that belongs to span$\{\mathbf{u}, \mathbf{v}\}$ (not in the same direction of $\mathbf{u}$ or $\mathbf{v}$).

   By definition of spanning set, any linear combination of $\mathbf{u}$ and $\mathbf{v}$ belongs to span$\{\mathbf{u}, \mathbf{v}\}$. For instance for $c_1 = 2$ and $c_2 = -1$,

$$c_1\mathbf{u} + c_2\mathbf{v} = 2\mathbf{u} - \mathbf{v} = 2\begin{bmatrix} -4 \\ 5 \\ 5 \end{bmatrix} - \begin{bmatrix} -4 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ 3 \end{bmatrix} \in \text{span}\{\mathbf{u}, \mathbf{v}\}.$$

(c) (7 points) (WebWork) Given the augmented matrices

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & -5 \\ 1 & 2 & -1 & 8 \\ 1 & 1 & 0 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right],$$

that is, they are row equivalent. Write $\begin{bmatrix} -5 \\ 8 \\ 4 \end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

We need to find scalars c_1, c_2 and c_3 such that

$$\begin{bmatrix} -5 \\ 8 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

This vector equation yields a linear system of equations (with unknowns c_1, c_2 and c_3) whose augmented matrix is given by the first one. It is also given that this augmented matrix is row equivalent to the second augmented matrix which is in reduced row echelon form. Therefore,

$$c_1 = 1, \quad c_2 = 3, \quad c_3 = -1.$$

Hence,

$$\begin{bmatrix} -5 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

2. (28 points) Choose the correct answer.

(i) Let A be a square matrix. Then $A - A^T$ is [symmetric / [✓ skew-symmetric] / none].

(ii) Let A, B and C are matrices such that A is nonzero, the operation $A(B - C)$ is defined and $A(B - C) = 0$. Then, is it necessarily true that $B = C$? [Yes / [✓ No]]

(iii) Let A and B be upper triangular matrices of same size. Then, is it true that their product is also upper triangular? [[✓ Yes] / No]

(iv) Suppose that a square matrix A can be row reduced to the identity matrix. Then, is it true that A is invertible? [[✓ Yes] / No]

(v) Suppose that columns of a square matrix A forms a set of linearly dependent vectors. Is it true that A is invertible? [Yes / [✓ No]]

(vi) Let E be the composition of a sequence of elementary matrices. Then, is it true that the system $Ex = 0$ has a unique solution? [[✓ Yes] / No]

(vii) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis set for a vector space V. If we add another vector, $\mathbf{v}_4 \in V$, to the set, will we still have a basis for V? [Yes / [✓ No]]

3. (Video Lecture) Let

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -7 & -5 \\ 6 & 8 & 2 \end{bmatrix}.$$

In this question, you are supposed to find LU-decomposition of the matrix A by employing the steps below.

(a) (6 points) Following elementary row operations, if applied sequentially, converts the matrix A to a row echelon form:

$$E_1 : 2R_1 + R_2 \to R_2, \quad E_2 : -3R_1 + R_3 \to R_3, \quad E_3 : 4R_2 + R_3 \to R_3.$$

In this first part of the question, write the elementary matrices associated to these row operations.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

(b) (7 points) Next, find $E = E_3E_2E_1$ and then find $U = EA$.
(Note: E_1, E_2 and E_3 are the operations to convert A to a row echelon form. Therefore, U must be an upper triangular matrix. Check yourself! If this is not the case for you, you probably make a calculation mistake. Correct your calculations.)

Using matrix multiplication, we get

$$E = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 1 \end{bmatrix}.$$

Then, again by matrix multiplication

$$U = EA = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 \\ -4 & -7 & -5 \\ 6 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}.$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

(c) (8 points) Find E^{-1}.
(Note: Since E is lower triangular, then E^{-1} must be lower triangular as well. Check yourself! If E^{-1} you found is not lower triangular, you probably make a calculation mistake. Correct your calculations.)

By reversing the row operations of the given elementary row operations, observe that $E_1^{-1} : -2R_1 + R_2 \to R_2$, $E_2^{-1} : 3R_1 + R_3 \to R_3$ and $E_3^{-1} : -4R_2 + R_3 \to R_3$. Therefore,

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

Using the property $(AB)^{-1} = B^{-1}A^{-1}$, we can write $E^{-1} = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$. Hence

$$E^{-1} = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}.$$

You can also find E^{-1} using Gauss-Jordan elimination

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 5 & 4 & 1 & 0 & 0 & 1. \end{array}\right] \longrightarrow \text{Gauss-Jordan} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -4 & 1 \end{array}\right]$$

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}$$

(d) (1 point) Express A as a factor of a lower triangular matrix and an upper triangular matrix.
(Hint: If $EA = U$, then $A = E^{-1}U$, where E^{-1} is lower triangular from part (c) and U is upper triangular from part (b).)

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -7 & -5 \\ 6 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

4. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & 1 & 8 \end{bmatrix}.$$

(a) (15 points) Find a row echelon form, R, for the matrix A.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & 1 & 8 \end{bmatrix} \xrightarrow{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 2 & 1 & 8 \end{bmatrix} \xrightarrow{-2R_1+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -3 & 2 \end{bmatrix} \xrightarrow{3R_2+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -4 \end{bmatrix}.$$

$$R = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) (10 points) Based on your result in part (a), what matrix relates R to the original matrix A? That is, find the matrix E such that $EA = R$.

Let E_j, $j = 1, 2, 3$ be the elementary matrix for the $j-$th elementary row operation performed in part (a). Then,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1, \end{bmatrix}$$

and A is related to the row echelon form R via the composition $E = E_3E_2E_1$. Hence,

$$E = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix}.$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix}$$

5. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}.$$

(a) (15 points) Show that whether $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent or independent.

Linear dependence or independence of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are determined by whether the vector equation

$$c_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} = 0$$

can be satisfied with coefficients c_1, c_2, c_3 that are not all zero. This vector equation leads to system of equations, with unknowns c_1, c_2 and c_3, whose augmented matrix is in the following form

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 0 \\ 1 & -1 & 6 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right].$$

Performing elementary row operations, we obtain following row echelon form

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 0 \\ 1 & -1 & 6 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 6 & 0 \\ 2 & 3 & 2 & 0 \\ 3 & 4 & 4 & 0 \end{array}\right] \xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 6 & 0 \\ 0 & 5 & -10 & 0 \\ 0 & 7 & -14 & 0 \end{array}\right] \xrightarrow{-7R_2+5R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 6 & 0 \\ 0 & 5 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We infer from the row echelon form that there exists infinitely many solutions, which guarantees the existence of some nontrivial solutions. Hence, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent.

(b) (10 points) Is it true that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for $\mathbb{R}^3$? Explain in words.

The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ consists of linearly dependent vectors, hence it cannot be a basis for $\mathbb{R}^3$.