02.05.2025

İzmir Institute of Technology
MSE 222 Applied Mathematics for Materials Science and Engineering, Spring 2025
Midterm II – Solution Key

Name: ______________________________

Student ID: ______________________________

Duration: 105 Minutes

Grade Table

Question:	1	2	3	4	5	Total
Points:	25	35	20	20	20	120
Score:						

1. (a) (10 points) (WebWork) Let $A = \begin{bmatrix} 0 & 2 & -3 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}$.

   (i) Find the determinant of A.

   Cofactor expansion with respect to first column yields

$$\det(A) = 1 \times (-1)^{3+1} \times \det\left(\begin{bmatrix} 2 & -3 \\ 1 & 3 \end{bmatrix}\right) = 9.$$

   $\det(A) = 9$

   (ii) Find the adjoint of A.

   Cofactors are

$$\begin{aligned} A_{1,1} &= -5, A_{1,2} = 3, A_{1,3} = -1, \\ A_{2,1} &= -8, A_{2,2} = 3, A_{2,3} = 2, \\ A_{3,1} &= 9, A_{3,2} = 0, A_{3,3} = 0. \end{aligned}$$

   Hence, the adjoint matrix is

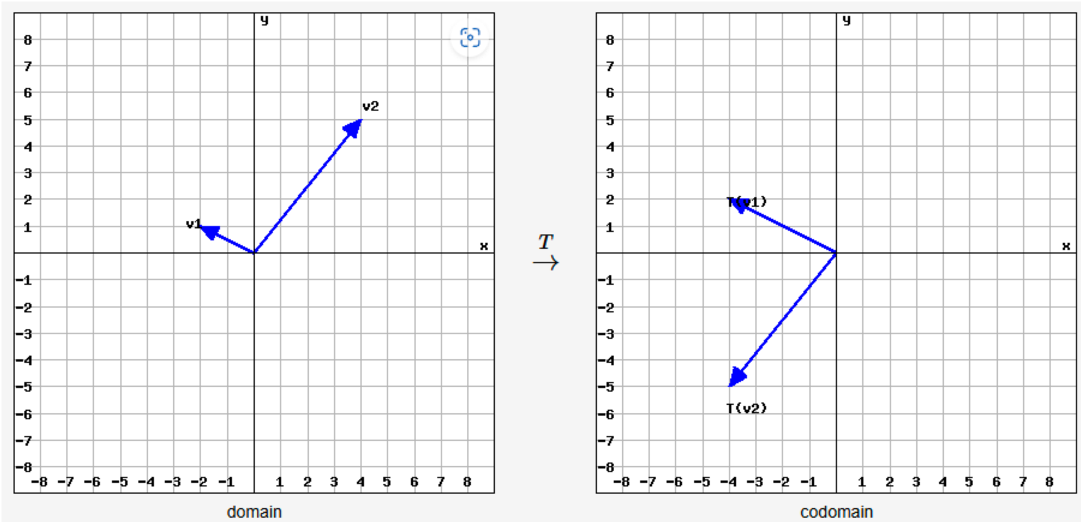
$$\mathrm{Adj}(A) = \begin{bmatrix} -5 & -8 & 9 \\ 3 & 3 & 0 \\ -1 & 2 & 0 \end{bmatrix}.$$

   (iii) Find the inverse of A.

   Inverse is given by $\frac{1}{\det(A)}\mathrm{Adj}(A)$. Hence, the inverse is

$$A^{-1} = \frac{1}{9}\begin{bmatrix} -5 & -8 & 9 \\ 3 & 3 & 0 \\ -1 & 2 & 0 \end{bmatrix}.$$

(b) (10 points) (WebWork) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. The figure shows where T maps the vectors v_1 and v_2 from the domain. With this limited information, answer the following questions.



(i) Explain (with at most 3 sentences) why v_1 and v_2 are eigenvectors for T. Then, write their corresponding eigenvalues.

$v_1 \mathbin{/\!/} T(v_1)$ and $v_2 \mathbin{/\!/} T(v_2)$ which are nonzero. Thus, v_1 and v_2 are eigenvalues for T.

$Tv_1 = 2v_1$

$Tv_2 = -1v_2$

(ii) If possible, find another eigenvector for T parallel to v_1 but not equal to v_1. If it is not possible, write "*DNE*".

Any scalar multiple of v_1 is again an eigenvector for T. So, for instance, we can take $2v_1 = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

(iii) How many nonzero vectors are eigenvectors with eigenvalue -2? [$\boxed{0}$ / 1 / 2 / 3 / 4 / infinitely many / cannot be determined] Explain your answer (with at most 3 sentences).

$T : \mathbb{R}^2 \to \mathbb{R}^2$. So, there exists at most two eigenvalues, which are $\lambda_1 = 2$ and $\lambda_2 = -1$. Hence -2 cannot be an eigenvalue.

(iv) Suppose that v is another vector that is neither parallel to v_1 nor parallel to v_2. Can v be an eigenvector for T? [yes / $\boxed{\text{no}}$ / cannot be determined] Explain your answer (with at most 3 sentences).

$T : \mathbb{R}^2 \to \mathbb{R}^2$ and we are already given two linearly independent eigenvectors. Therefore, there cannot be a third eigenvector which is not linearly independent to both v_1 and v_2.

(v) What is the degree of characteristic polynomial of T. Explain (with at most 3 sentences).

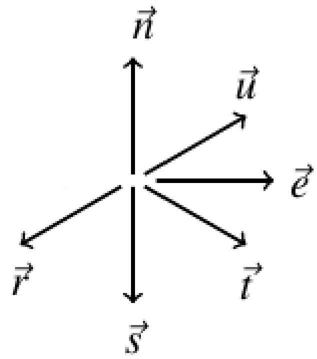
$T : \mathbb{R}^2 \to \mathbb{R}^2$. So, degree of $P_T(\lambda)$ is 2.

(vi) Write a characteristic polynomial of T.

$\lambda_1 = 2$ and $\lambda_2 = -1$. Then,

$$P_T(\lambda) = (\lambda - 2)(\lambda + 1) = \lambda^2 - \lambda - 2.$$

(c) (5 points) (WebWork) Several unit vectors in the $xy-$plane are shown in the figure.



Using the geometric definition of the inner product, are the following inner products positive, negative or zero?

(i) $\langle \vec{e}, \vec{s} \rangle$ [positive / negative / $\boxed{\text{zero}}$]

(ii) $\langle \vec{s}, \vec{t} \rangle$ [$\boxed{\text{positive}}$ / negative / zero]

(iii) $\langle \vec{r}, \vec{s} \rangle$ [$\boxed{\text{positive}}$ / negative / zero]

(iv) $\langle \vec{n}, \vec{e} \rangle$ [positive / negative / $\boxed{\text{zero}}$]

(v) $\langle \vec{n}, \vec{t} \rangle$ [positive / $\boxed{\text{negative}}$ / zero]

2. Answer the following questions. Write an explanation (with at most 3 sentences), if required.

(a) (5 points) If A is a 5×5 matrix of rank 4, is it true that A has an eigenvalue 0? Explain.

True. Rank$(A) = 4 < 5$ implies only 4 of the row vectors of A are linearly independent and the remaining fifth one is linearly dependent. This implies $\det(A) = 0$.

(b) (5 points) Let A be a nonsingular 3×3 matrix. Is it true that row vectors of A form a basis for $\mathbb{R}^3$? Explain.

Yes.

$$A \text{ is nonsingular} \quad \Rightarrow \quad \det(A) \neq 0 \quad \Rightarrow \quad \text{row vectors of } A \text{ are linearly independent.}$$

Since the number of row vectors are three (which is dimension of $\mathbb{R}^3$), they form a basis for $\mathbb{R}^3$.

(c) (5 points) Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation matrix that rotates the vectors of $\mathbb{R}^2$ by π radians counterclockwise. Without making any calculations, write the space of eigenvectors of A. Explain your answer. (Hint: Think geometrically, you do not need any calculations to answer this question.)

If we rotate a vector in $\mathbb{R}^2$ by π radians, then the rotated vector and the original vector remains parallel, i.e., one is a scalar multiple of another. This is true for all vectors of $\mathbb{R}^2$. Hence, all vectors of $\mathbb{R}^2$ are eigenvectors for A.

(d) (5 points) Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection matrix that projects every vector of $\mathbb{R}^3$ onto the $xy-$plane. Give an example to an eigenvector for P. According to your answer, what is the associated eigenvalue? Explain your answer (you do not need to perform any calculations to answer this question).

P projects every vector in the $xy-$plane onto itself without scaling its length. Therefore, every vector in the $xy-$plane is an eigenvector for P with an eigenvalue 1.

(e) (5 points) Let A be a 5×5 matrix with rank equal to 3 and $\mathbf{b}$ be any vector in $\mathbb{R}^5$. Does the system $A\mathbf{x} = \mathbf{b}$ have a solution or not? If $A\mathbf{x} = \mathbf{b}$ has a solution, is it the unique one or there exists infinitely many solutions? Explain.

$\text{Rank}(A) = 3 < 5$, so existence of a solution depends on $\mathbf{b}$:

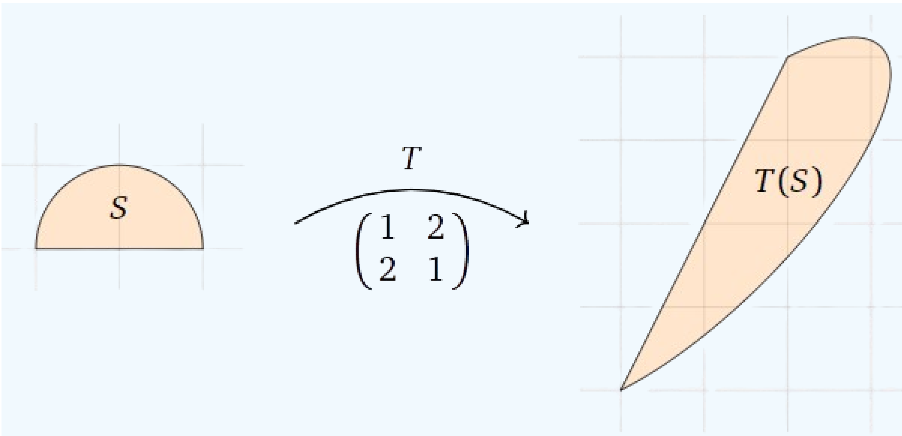
$$\begin{cases} \text{infinitely many solution,} & \text{if } \text{Rank}(A) = \text{Rank}[A \,|\, \mathbf{b}], \\ \text{no solution,} & \text{if } \text{Rank}(A) < \text{Rank}[A \,|\, \mathbf{b}]. \end{cases}$$

Since $\text{Rank}(A) = 3 < 5$, we are sure that if a solution exists, it is not unique.

(f) (5 points) Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be the composition of three rotation matrices in $\mathbb{R}^3$ each of them counterclockwise direction: rotation in the $xy-$plane by α radians, rotation in the $xz-$plane by β radians and rotation in the $yz-$axis by γ radians. What is $\det(A)$? Give reasoning to your answer (you do not need to perform any calculations to answer this question).

Rotation operation does not scale elements. Therefore, $\det(A) = 1$.

(g) (5 points) Let S be the half-circle of radius 1. Define the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. What is the area of $T(S)$?



$\text{Area}(S) = \frac{\pi}{2}$ and $|\det(T)| = 3$. Hence,

$$\text{Area}(T(S)) = |\det(T)| \times \text{Area}(S) = \frac{3\pi}{2}.$$

3. Let $\mathbb{P}_3$ denote the space of polynomials having real coefficients with degree less than or equal to 3 whose basis is given by the set $S = \{1, x, x^2, x^3\}$. Below, we give a different representation of polynomials of $\mathbb{P}_3$ in vector form:

   Let $ax^3 + bx^2 + cx + d \in \mathbb{P}_3$, then we give its vector representation as

$$\begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} \in \mathbb{R}^4.$$

   For example vector representation of the polynomial $p(x) = -3x^3 + x - 2$ (which is in $\mathbb{P}_3$) is

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ -3 \end{bmatrix}.$$

   (a) (4 points) Write each of the basis elements $S = \{1, x, x^2, x^3\}$ of $\mathbb{P}_3$ in vector form as described above.

   Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be the set of vectors that are the vector representations of elements of S. Then,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

   (b) (8 points) Consider the differentiation operator $\frac{d}{dx}$ with domain $\mathbb{P}_3$ and range $\mathbb{P}_2$:

$$\frac{d}{dx} : p \in \mathbb{P}_3 \to p' \in \mathbb{P}_2.$$

   Using the vector description of the polynomials given above, find the associated differentiation operator, D, in matrix form.

   $\frac{d}{dx}(ax^3 + bx^2 + cx + d) = 3ax^2 + 2bx + c$. Therefore, using the vector representation, D is a transformation defined on $\mathbb{R}^4 \to \mathbb{R}^3$ such that

$$D\left(\begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}\right) = \begin{bmatrix} c \\ 2b \\ 3a \end{bmatrix}.$$

   Then, D maps vector representation of the basis elements $1, x, x^2, x^3$ to the following vectors of $\mathbb{R}^3$:

$$D\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad D\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad D\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad D\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

   Hence, the matrix representation of D is

$$D = \begin{bmatrix} D(\mathbf{v}_1) & D(\mathbf{v}_2) & D(\mathbf{v}_3) & D(\mathbf{v}_4) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

(c) (8 points) Find null(D).

Our task is to find $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$ such that $D(\mathbf{u}) = \mathbf{0} \in \mathbb{R}^3$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow u_2 = 0, u_3 = 0, u_4 = 0.$$

Set $u_1 = s \in \mathbb{R}$. Then, the vectors of the form $\mathbf{u} = \begin{bmatrix} s \\ 0 \\ 0 \\ 0 \end{bmatrix}$ are in the null space of D, i.e.,

$$\text{Null}(D) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Remark. Null space of the differentiation operator $\frac{d}{dx}$ is constant polynomials which are represented by the vectors $\begin{bmatrix} s & 0 & 0 & 0 \end{bmatrix}^T$, $s \in \mathbb{R}$. So the result in this part is consistent in the sense that null space of $\frac{d}{dx}$ and D coincides in this sense.

4. Consider the linear system of equations

$$\begin{aligned} x_1 + x_2 &= 3, \\ -3x_1 + 2x_3 &= 0, \\ x_2 - 2x_3 &= 0. \end{aligned}$$

(a) (8 points) Without solving the system and using the determinants, show that this system has a unique solution.

Let A be the coefficients matrix of the given linear system. We need to show that A is nonsingular by showing that $\det(A) \neq 0$:

$$\det(A) = \det\left(\begin{bmatrix} 1 & 1 & 2 \\ -3 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix} \right) = -14.$$

(b) (12 points) Next, solve the system using the Cramer's rule.

Let A_j, $j = 1, 2, 3$ be the matrix that is formed by replacing the $j-th$column with the right hand side vector. Then, $\det(A_1) = \det\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix} \right) = -6$, $\det(A_2) = \det\left(\begin{bmatrix} 1 & 3 & 2 \\ -3 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \right) = -18$, $\det(A_3) = \det\left(\begin{bmatrix} 1 & 1 & 3 \\ -3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right) = -9$. Consequently, by the Cramer's rule, we get

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{3}{7}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{9}{7}, \quad x_3 = \frac{\det(A_3)}{\det(A)} = \frac{9}{14}.$$

5. Consider the matrix

$$A = \begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}$$

(a) (10 points) Find eigenvalues and eigenvectors of A.

Step 1: Eigenvalues. Find λ such that $\det(A - \lambda I) = 0$.

$$\det\left(\begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 5-\lambda & 6 \\ -2 & -2-\lambda \end{bmatrix}\right)) = 0 \Rightarrow (5-\lambda)(-2-\lambda) + 12 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$$

This implies $\lambda_1 = 1$, $\lambda_2 = 2$.

Step 2: Eigenvectors. Let $\mathbf{v}_1 = \begin{bmatrix} v_1^1 \\ v_1^2 \end{bmatrix}$ be the eigenvector for the eigenvalue $\lambda_1 = 1$. We need to solve the system $(A - \lambda_1 I)\mathbf{v}_1 = 0$ to find $\mathbf{v}_1$.

$$(A - \lambda_1 I)\mathbf{v}_1 = 0 \Rightarrow \begin{bmatrix} 4 & 6 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Set $3s = v_1^1$, $s \in \mathbb{R}$. Then $v_1^2 = -2s$. If we choose $s = 1$, then corresponding eigenvector is

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Let $\mathbf{v}_2 = \begin{bmatrix} v_2^1 \\ v_2^2 \end{bmatrix}$ be the eigenvector for the eigenvalue $\lambda_1 = 2$. Similarly, we solve $(A - \lambda_2 I)\mathbf{v}_2 = 0$

$$(A - \lambda_1 I)\mathbf{v}_1 = 0 \Rightarrow \begin{bmatrix} 3 & 6 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} v_2^1 \\ v_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

and find $v_2^1 = 2s$, $v_2^2 = -s$, $s \in \mathbb{R}$. Choosing $s = 1$ yields

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

(b) (10 points) (Video Lecture) Find a matrix P such that $P^{-1}AP$ becomes a diagonal matrix.

We form the matrix P by setting the first column as $\mathbf{v}_1$ and second column as $\mathbf{v}_2$

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}.$$

Then A is diagonalizable in the sense that $P^{-1}AP$ becomes a diagonal matrix with diagonal entries λ_1 and λ_2:

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$