

İzmir Institute of Technology
MSE 222 Applied Mathematics for Materials Science and Engineering, Spring 2025
Quiz II – Solution Key

Name: _____

Student ID: _____

Duration: 40 Minutes

Grade Table

Question:	1	2	3	4	5	6	Total
Points:	20	15	15	15	20	15	100
Score:							

1. (20 points) Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 2 & -3 & 0 & 2 \\ 2 & -1 & 4 & -1 \\ -4 & 6 & 0 & -4 \end{bmatrix}.$$

$\text{Col}(A) = \text{Row}(A^T)$. So, let us employ the following row operations for the transpose matrix A^T :

$$A^T = \begin{bmatrix} 2 & 2 & -4 \\ -3 & -1 & 6 \\ 0 & 4 & 0 \\ 2 & -1 & -4 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 \rightarrow R_1} \begin{bmatrix} 1 & 1 & -2 \\ -3 & -1 & 6 \\ 0 & 4 & 0 \\ 2 & -1 & -4 \end{bmatrix} \xrightarrow{\begin{matrix} 3R_1 + R_2 \rightarrow R_2 \\ -2R_1 + R_4 \rightarrow R_4 \end{matrix}} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & -3 & 0 \end{bmatrix}.$$

Second, third and fourth rows are linearly dependent, whereas the first row is linearly independent with others. Hence, a basis for the row space of A^T , that is the basis for the column space of A is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

2. (15 points) Suppose that A is an $n \times m$ matrix of rank 4, the nullity of A is 3 and the column space of A is a subspace of $\mathbb{R}^7$. Find the dimensions of A .

Given that, $\text{Rank}(A) = 4$ and $\dim(\text{Null}(A)) = 3$. So, by the Rank-Nullity theorem, $\text{Rank}(A) + \dim(\text{Null}(A)) = 4 + 3 = 7$ is the number of rows of A . $\text{Col}(A)$ is a subspace of $\mathbb{R}^7$. Therefore, number of columns of A is 7. Hence,

A has 7 rows and 7 columns.

3. Suppose that A is a 3×7 matrix that has an echelon form with two zero rows. Find the dimension of the row space of A , the dimension of the column space of A and the dimension of the null space of A .

Echelon form of A has two zero rows. So, number of linearly independent row vectors are $3 - 2 = 1$ which gives $\dim(\text{Row}(A)) = \dim(\text{Col}(A)) = 1$. Hence, $\text{Rank}(A) = 1$. Given that number of columns of A is 7. Therefore, by the Rank-Nullity theorem $\dim(\text{Null}(A)) = 7 - 1 = 6$. Hence, we have the following results.

- (a) (5 points) The dimension of the row space of A is 1.
 (b) (5 points) The dimension of the column space of A is 1.
 (c) (5 points) The dimension of the null space of A is 6.
4. (15 points) Find a basis for the null space of the matrix

$$A = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

$A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \in \mathbb{R}^3$, yields

$$\begin{aligned} 3x_1 - 2x_2 &= 0, \\ x_1 + x_2 + x_3 &= 0. \end{aligned}$$

Set $x_1 = 2s$, where $s \in \mathbb{R}$ is a parameter. Then, $x_2 = 3s$ and $x_3 = -5s$. Consequently, solutions to $A\mathbf{x} = \mathbf{0}$ is of the form $\mathbf{x} = s \begin{bmatrix} 2 & 3 & -5 \end{bmatrix}^T$. Hence,

$$\text{Null}(A) = \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix} \right\}.$$

5. (20 points) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation that first reflects the points through the x -axis, then reflect the points through the line $y = x$ and then rotate the points by a $\frac{\pi}{2}$ radians counterclockwise. Find the standard matrix A for T .

Reflection through the x -axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Reflection through the line $y = x$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Rotation by $\frac{\pi}{2}$ radians counterclockwise: $\begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Thus,

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Alternative way.

$$\begin{aligned} \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} &\xrightarrow{\text{reflection through } x\text{-axis}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{\text{reflection through the line } y = x} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{\text{rotation counterclockwise by } \theta = \frac{\pi}{2} \text{ radians}} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \Rightarrow T\mathbf{e}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} &\xrightarrow{\text{reflection through } x\text{-axis}} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \xrightarrow{\text{reflection through the line } y = x} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \xrightarrow{\text{rotation counterclockwise by } \theta = \frac{\pi}{2} \text{ radians}} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow T\mathbf{e}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{ Hence,} \end{aligned}$$

$$T = [T\mathbf{e}_1 \quad T\mathbf{e}_2] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

6. (15 points) Consider the following Gauss elimination:

$$A \rightarrow \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} A \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{E_2} E_1 A \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}_{E_3} E_2 E_1 A \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_4} E_3 E_2 E_1 A \rightarrow \begin{bmatrix} -4 & 3 & 5 \\ 0 & 1 & -8 \\ 0 & 0 & 3 \end{bmatrix}$$

What is the determinant of A ?

$$\det(E_1) = -1, \det(E_2) = 3, \det(E_3) = 1, \det(E_4) = \frac{1}{4} \text{ and } \det\left(\begin{bmatrix} -4 & 3 & 5 \\ 0 & 1 & -8 \\ 0 & 0 & 3 \end{bmatrix}\right) = (-4) \times 1 \times 3 = -12. \text{ Then,}$$

$$\begin{aligned} -12 &= \det\left(\begin{bmatrix} -4 & 3 & 5 \\ 0 & 1 & -8 \\ 0 & 0 & 3 \end{bmatrix}\right) = \det(E_4 E_3 E_2 E_1 A) \\ &= \det(E_4) \det(E_3) \det(E_2) \det(E_1) \det(A) \\ &= -\frac{3}{4} \det(A) \Rightarrow \det(A) = 16. \end{aligned}$$