24.07.2025

İzmir Institute of Technology
Math 255 Differential Equations, Summer 2025
Midterm I - Solution Key

Name: ______________________________

Student ID: ______________________________

Department: ______________________________

Duration: 120 Minutes

Grade Table

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

---

1. (WebWork)

   (a) (12 points) Determine whether each first-order differential equation is separable, linear, both or neither.

   (i) $\frac{dy}{dx} + e^x y^2 = x^2 y^2$. [$\boxed{\text{Separable}}$ / Linear / Both / Neither]

   (ii) $y + \sin x = x^3 \frac{dy}{dx}$. [Separable / $\boxed{\text{Linear}}$ / Both / Neither]

   (iii) $y \ln x - x^2 y = x \frac{dy}{dx}$. [Separable / Linear / $\boxed{\text{Both}}$ / Neither]

   (iv) $\frac{dy}{dx} + \cos y = \tan x$. [Separable / Linear / Both / $\boxed{\text{Neither}}$]

   (b) (8 points) Fill the following table: (i) identify the independent variable, (ii) identify the dependent variable, (iii) give the order of each differential equation and (iv) state whether the equation is linear or nonlinear.

Equation	(i) Independent	(ii) Dependent	(iii) Order	(iv) Linear/Nonlinear
$y' = y - x^2$	x	y	1	Linear
$xy' = 2y$	x	y	1	Linear
$\dfrac{d^3z}{ds^3} = \sqrt{1 + \left(\frac{dz}{ds}\right)^2}$	s	z	3	Nonlinear
$\dfrac{d^2x}{dt^2} + 5x = e^{-x}$	t	x	2	Nonlinear

2. (20 points) (WebWork) Solve the initial value problem

$$xy' + y = -9xy^3, \quad y(2) = 1.$$

Given equation is a Bernoulli differential equation with $n = 3$. So, we make the following substitution

$$\begin{aligned} z &= y^{1-3} = y^{-2}, \\ z' &= -2y^{-3}y'. \end{aligned} \qquad (\star)$$

Dividing both sides of the equation by y^3, we get

$$xy'y^{-3} + y^{-2} = -9x.$$

Then, we use the substitution $(\star)$ and write the equation in the function z as

$$-\frac{x}{2}z' + z = -9x.$$

This is a first-order linear equation in standard form as

$$z' - \frac{2}{x}z = 18,$$

with integration factor $\mu(x) = e^{\int\left(-\frac{2}{x}\right)dx} = x^{-2}$. We solve this equation first by multiplying both sides by $\mu(x) = x^{-2}$ and then by integrating as

$$\begin{aligned} z' - \frac{2}{x}z = 18 &\overset{\times x^{-2}}{\Longrightarrow} x^{-2}z' - 2x^{-3}z = 18x^{-2} \\ &\Longrightarrow \frac{d}{dx}\left(x^{-2}z\right) = 18x^{-2} \\ &\overset{\int dx}{\Longrightarrow} x^{-2}z = -18x^{-1} + C, \quad C \in \mathbb{R} \\ &\Longrightarrow z(x) = -18x + Cx^2 \end{aligned}$$

Using the substitution $z = y^{-2}$ back, we find the general solution of the $y-$equation in implicit form as

$$\frac{1}{y^2(x)} = -18x + Cx^2.$$

Finally, employing the initial condition, we find C as

$$\begin{aligned} \frac{1}{y^2(x)} = -18x + Cx^2 &\overset{y(2)=1}{\Longrightarrow} 1 = -18 \times 2 + C \times 2^2 \\ &\Longrightarrow C = \frac{37}{4}. \end{aligned}$$

Hence, solution to the initial value problem is

$$\boxed{\frac{1}{y^2(x)} = -18x + \frac{37x^2}{4}}.$$

3. (20 points) Use the change of dependent variable $z(x) = y(x) - 2x$ to find the general solution to the following differential equation.

$$\frac{dy}{dx} = y - 2x + 2 - (2x - y)^{-1}$$

Let $z(x) = y(x) - 2x$, then $z'(x) = y'(x) - 2$. Substituting into the equation yields

$$z' + 2 = z + 2 - (-z)^{-1}$$

or

$$z' = z + z^{-1}.$$

This is a separable equation and can be written as

$$dx - \frac{z}{z^2 + 1} dz = 0$$

with $m(x) = 1$ and $n(z) = \frac{z}{z^2+1}$. Anti-derivatives of $m(x)$ and $n(z)$ in their repsective variables are

$$\int m(x)dx = \int 1dx = x,$$
$$\int n(z)dz = \int \frac{z}{z^2 + 1} dz = \frac{1}{2} \ln |z^2 + 1|.$$

It follows that

$$x - \frac{1}{2} \ln \left|z^2 + 1\right| = C, \quad C \in \mathbb{R}.$$

Finally, we replace z by $y - 2x$ to obtain the general solution as

$$\boxed{x - \frac{1}{2} \ln \left|(y - 2x)^2 + 1\right| = C}.$$

Note: $z-$equation can be classified as Bernoulli differential equation with $n = -1$. Hence, it can also be solved by applying the strategy we introduced for solving Bernoulli differential equations.

4. (20 points) Solve one of the following questions.

(a) The population of mosquitoes in a certain area increases at a rate proportional to its square with the constant rate 2. There are 200,000 mosquitoes in the area initially. Determine the population of mosquitoes in the area at any time t.

Let t represents the time and $P(t)$ be the population of the mosquitoes at any time t. Initially, there are 200,000 mosquitoes in the area. Therefore, $P(0) = 200000$.

- Rate of increase: Proportional to the square of the population: kP^2 for some constant rate k.
- Proportionality constant: The constant rate is given by 2. Therefore $k = 2$.

Consequently, the governing initial value problem is

$$\frac{dP}{dt} = 2P^2, \quad P(0) = 200000,$$

which is a separable equation. We rewrite the equation as

$$2dt - \frac{1}{P^2}dP = 0,$$

and then solve it to obtain the general solution as

$$2t + \frac{1}{P(t)} = C.$$

Employing the initial condition $P(0) = 200000$ yields $C = \frac{1}{200000}$. Hence, the population of mosquitoes as a function of time is explicitly given by

$$\boxed{P(t) = \frac{1}{\frac{1}{200000} - 2t}}.$$

(b) A tank initially contains 120 liters of pure water. A mixture with a salt concentration of 10 grams per liter enters the tank at a rate of 2 liters per minute, and the well-stirred mixture leaves the tank at the same rate. Determine the amount of salt in the tank at any time t.

Let t represents the time in minutes and $y = y(t)$ be the amount of salt in the tank measured in grams. Initially, the tank contains pure water, i.e., the initial amount of salt is 0. Therefore, $y(0) = 0$.

- Rate in: Salt enters to the tank at a rate of 10 gr/liter $\times$ 2 liter/min = 20 gr/min.
- Rate out: Observe that the volume of the mixture remains same and 120 liters for all $t > 0$. Therefore, salt leaves the tank at a rate of $y(t)/120$ gr/liter $\times$ 2 liter/min = $\frac{y(t)}{60}$ gr/min.

Consequently, the governing initial value problem is

$$\frac{dy}{dt} = 20 - \frac{1}{60}y, \quad y(0) = 0,$$

which is first-order and linear with integration factor $\mu(t) = e^{t/60}$. Multiplying both sides of the equation by $\mu(t)$ and then integrating yields

$$\frac{dy}{dt} + \frac{1}{60}y = 20 \overset{\times e^{t/60}}{\Longrightarrow} \frac{d}{dt}\left(y(t)e^{t/60}\right) = 20e^{t/60}$$
$$\overset{\int dt}{\Longrightarrow} y(t)e^{t/60} = 1200e^{t/60} + C.$$

Employing the initial condition $y(0) = 0$, we get $C = -1200$. Hence, the amount of salt in the tank as a function of time is given by

$$\boxed{y(t) = 1200 - 1200e^{-t/60}}.$$

5. Consider the differential equation

$$(ye^{2xy}+x)dx + bxe^{2xy}dy = 0.$$

(a) (6 points) Find the value of b for which the given equation is exact.

(b) (14 points) Find the general solution to the equation using that value of b.

(a) Let $M(x,y) = ye^{2xy} + x$ and $N(x,y) = bxe^{2xy}$. Then,

$$\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial y}\left(ye^{2xy}+x\right) = e^{2xy} + 2xye^{2xy}$$

and

$$\frac{\partial}{\partial x}N(x,y) = \frac{\partial}{\partial x}\left(bxe^{2xy}\right) = be^{2xy} + 2xybe^{2xy}.$$

In order for the exactness criterion $\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$ to be hold, we must have

$$\boxed{b=1}.$$

(b) We take $b=1$ so that the given equation is an exact differential equation. This yields existence of a function $\psi(x,y)$ so that

$$\frac{\partial}{\partial x}\psi(x,y) = M(x,y) \quad \text{and} \quad \frac{\partial}{\partial y}\psi(x,y) = N(x,y)$$

holds true. To find ψ, and therefore the general solution, we follow the two step strategy given below:

- Step 1: We integrate $M(x,y)$ with respect to x:

$$\psi(x,y) = \int M(x,y)dx = \int \left(ye^{2xy}+x\right)dx = \frac{e^{2xy}}{2} + \frac{x^2}{2} + g(y). \quad (\star)$$

- Step 2: To determine $g(y)$, we differentiate $(\star)$ with respect to y and set the result $N(x,y)$:

$$\begin{aligned}\frac{\partial}{\partial y}\psi(x,y) = N(x,y) &\Longrightarrow \frac{\partial}{\partial y}\left(\frac{e^{2xy}}{2}+\frac{x^2}{2}+g(y)\right) = xe^{2xy}\\ &\Longrightarrow xe^{2xy} + g'(y) = xe^{2xy}\\ &\Longrightarrow g'(y) = 0.\end{aligned}$$

Hence, $g(y) = C$, $C \in \mathbb{R}$. It follows from $(\star)$ that the general solution is given implicitly by

$$\boxed{\psi(x,y) = \frac{e^{2xy}}{2} + \frac{x^2}{2} + C = 0}.$$