07.08.2025

İzmir Institute of Technology

Math 255 Differential Equations, Summer 2025

Midterm II – Solution Key

Name: ______________________________

Student ID: ______________________________

Department: ______________________________

Duration: 120 Minutes

Read the instructions below.

- This exam has five questions (verify).
- The second question is from WebWork. For this question, providing a fully correct answer without presenting any calculations is enough to get full credit.
- Other than the second question, you are expected to provide detailed calculations and explanations that clarifies the method of your solution. A correct answer without any calculations/explanations will not receive any credit. An incorrect answer including partially correct calculations will receive partial credit.
- Present your work in the empty sheets you are provided at the beginning of the exam. No extra sheets will be provided.
- You may not use any electronic device such as phones, smart watches etc., including calculators. Attempting to use any kind of electronic device as well as bringing an extra paper by you will be treated as cheating attempt.
- You may not leave the exam during the first 30 minutes.

Grade Table

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

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1. (20 points) Given a particular solution $y_1(t) = t$, find the general solution of the following differential equation

$$y' = 1 + t^2 - 2ty + y^2.$$

Given equation is first-order, nonhomogeneous and nonlinear with a quadratic nonlinearity. Therefore, it is a Riccati equation. We are also given a particular solution $y_1(t) = t$. So, we use the following change of dependent variable

$$y(t) = \frac{1}{z(t)} + t \quad \Longrightarrow \quad y'(t) = -\frac{z'(t)}{z^2(t)} + 1. \quad (\star)$$

Then, we substitute y and y' into the equation to obtain

$$-\frac{z'}{z^2} + 1 = 1 + t^2 - 2t\left(\frac{1}{z} + t\right) + \left(\frac{1}{z} + t\right)^2.$$

Simplifying the equation, we can write

$$-\frac{z'}{z^2} = \frac{1}{z^2}$$

or equivalently

$$z' = -1.$$

Direct integration with respect to t yields

$$z(t) = -t + C, \quad C \text{ is the integration constant.}$$

Finally, we use the substitution $(\star)$ back and deduce the general solution to the given differential equation as

$$\boxed{y(t) = \frac{1}{C - t} + t}.$$

Note: You can also use the substitution $y(t) = z(t) + t$ to convert the given differential equation to a Bernoulli equation and then solve it.

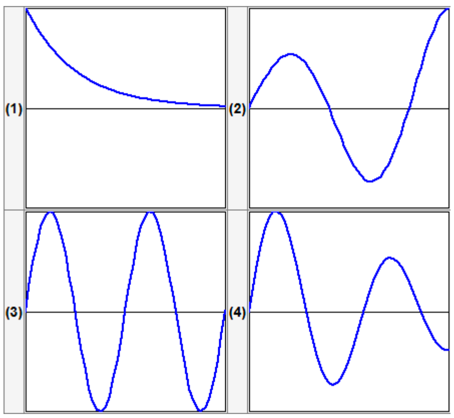
2. (a) (10 points) (WebWork) Match the graphs of solutions shown in the figure below with each of the differential equations below.

(A) $x'' + 4x = 0$: Graph 3

(B) $x'' - 4x = 0$: Graph 1

(C) $x'' - 2x' + 5x = 0$: Graph 2

(D) $x'' + 2x' + 5x = 0$: Graph 4



(b) (10 points) (WebWork) Match the following guess solutions y_p for the method of undetermined coefficients with the second-order nonhomogeneous linear equations below.

A. $y_p(x) = Ax^2 + Bx + C$

B. $y_p(x) = Ae^{2x}$

C. $y_p(x) = A\cos(2x) + B\sin(2x)$

D. $y_p(x) = (Ax + B)\cos(2x) + (Cx + D)\sin(2x)$

E. $y_p(x) = Axe^{2x}$

F. $y_p(x) = e^{3x}(A\cos(2x) + B\sin(2x))$

1. $\boxed{\text{E}}$ $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{2x}$

2. $\boxed{\text{A}}$ $\dfrac{d^2y}{dx^2} + 4y = -3x^2 + 2x + 3$

3. $\boxed{\text{C}}$ $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 20y = -3\sin(2x)$

4. $\boxed{\text{F}}$ $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 15y = e^{3x}\cos(2x)$

3. The motion of an object with mass $m > 0$ stretched to a spring with a spring constant $k > 0$ by taking into account the damping coefficient $b > 0$ is modeled by the initial value problem

$$\begin{cases} mu'' + bu' + ku = 0, \\ u(0) = u_0, \quad u'(0) = v_0, \end{cases}$$

where $u = u(t)$ is the displacement of the mass from the equilibrium in time t, u_0 is the initial displacement of the mass and v_0 is its initial velocity.

Suppose that a body with a mass 0.1 kg stretches a spring 0.2 m. The mass is set in motion from its equilibrium position with a downward velocity 1 m/s.

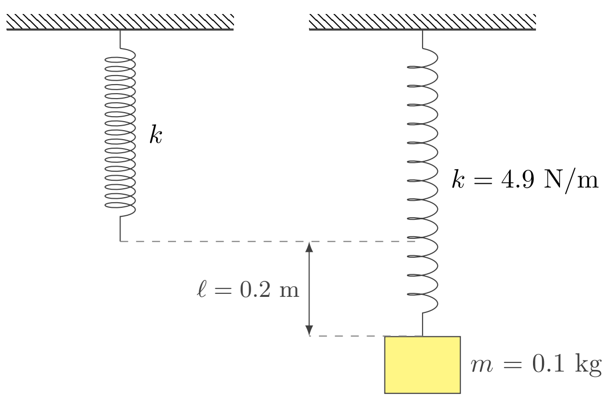
(a) (6 points) If there is no damping, then write the differential equation together with the initial conditions that describes the displacement, $u(t)$, of the mass in time t.

(b) (14 points) Find $u(t)$ by solving the initial value problem you wrote in part (a).

(Suppose that forces that cause the spring to stretch have positive sign and vice versa. The standard gravity $g = 9.8$ m/s^2.)

(a) Before the mass is set in motion, it stretches the spring by $\ell = 0.2$ meters and remains at rest. In this static situation, the gravitational force and the restoring force balance each other so that

$$mg - k\ell = 0 \quad \Longrightarrow \quad k = \frac{mg}{\ell} = \frac{0.1 \times 9.8}{0.2} = 4.9 \text{ N/m}.$$



The movement is initialized by the following:

- The mass is released from its equilibrium position. That means, the initial displacement is $u(0) = 0$m.
- The mass is given an initial velocity 1 m/s with a downward direction. Notice that the downward velocity causes the spring to stretch. Therefore, with a positive sign, the second initial condition is $u'(0) = +1$m/s.

In addition, the coefficients that are involved in the sought after differential equation are as following:

- The mass is given by $m = 0.1$kg.
- We consider the case where there is no damping. So $b = 0$.
- Stiffness of the spring is obtained as $k = 4.9$N/m above.

Consequently, the governing initial-value problem that describes the motion of the mass in time is

$$\begin{cases} 0.1u'' + 4.9u = 0, \quad t > 0, \\ u(0) = 0, \quad u'(0) = 1. \end{cases}$$

(b) The equation is a second-order homogeneous one with constant coefficients. Corresponding characteristic equation is

$$r^2 + 49 = 0.$$

with roots $r_{1,2} = \mp 7i$. Hence, the general solution is of the form

$$u_g(t) = c_1 \cos(7t) + c_2 \sin(7t).$$

Employing the first condition $u(0) = 0$, we get

$$\begin{aligned} u(t) = c_1 \cos(7t) + c_2 \sin(7t) \quad &\overset{t=0}{\Longrightarrow} \quad u(0) = c_1 \cos(0) + c_2 \sin(0) = 0 \\ &\Longrightarrow \quad c_1 = 0. \end{aligned}$$

The second initial condition yields

$$\begin{aligned} u(t) = c_2 \sin(7t) \quad &\Longrightarrow \quad u'(t) = 7c_2 \cos(7t) \\ &\overset{t=0}{\Longrightarrow} \quad u'(0) = 7c_2 \cos(0) = 1 \\ &\Longrightarrow \quad c_2 = \frac{1}{7}. \end{aligned}$$

Hence, the displacement of the mass from the equilibrium in time is given by

$$\boxed{u(t) = \frac{1}{7} \sin(7t)}.$$

4. (20 points) Find the general solution of the following differential equation

$$y'' + 3y' + 2y = \frac{1}{1 + e^t}.$$

Given differential equation is linear and nonhomogeneous. Therefore, the general solution is sum of complementary solution and a particular solution.

Step 1: Complementary solution. Homogeneous part is constant coefficient. So, we find the associated characteristic equation as

$$r^2 + 3r + 2 = 0$$

with roots $r_1 = -2$ and $r_2 = -1$. Therefore, two linearly independent solutions are $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-t}$, and the complementary solution is

$$y_c(t) = c_1 e^{-2t} + c_2 e^{-t}, \quad c_{1,2} \in \mathbb{R}.$$

Step 2: Particular solution. Note that the method of undetermined coefficients is not applicable to find a particular solution due to the structure of the nonhomogeneous part

$$g(t) = \frac{1}{1 + e^t}.$$

This leads us to apply the variation of parameters method. To this end, let us first find the Wronskian of the fundamental solutions

$$W(y_1(t), y_2(t)) = \det \begin{pmatrix} e^{-2t} & e^{-t} \\ -2e^{-2t} & -e^{-t} \end{pmatrix} = e^{-3t}.$$

In view of the variation of parameters method, the particular solution is given by $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ in which

$$\begin{aligned} u_1(t) &= -\int \frac{y_2(t)g(t)}{W(y_1(t), y_2(t))} dt \\ &= -\int \frac{e^{2t}}{1+e^t} dt, \quad \text{substitution: } s = e^t \Rightarrow ds = e^t dt \\ &= -\int \frac{s}{s+1} ds \\ &= -\int \left(1 - \frac{1}{s+1}\right) ds \\ &= -e^t + \ln(e^t + 1) \end{aligned}$$

and

$$\begin{aligned} u_2(t) &= \int \frac{y_1(t)g(t)}{W(y_1(t), y_2(t))} dt \\ &= \int \frac{e^t}{1+e^t} dt, \quad \text{substitution: } s = e^t \Rightarrow ds = e^t dt \\ &= \int \frac{1}{s+1} ds \\ &= \ln(e^t + 1). \end{aligned}$$

Hence, the particular solution is

$$\begin{aligned} y_p(t) &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ &= \left(-e^t + \ln(e^t+1)\right) e^{-2t} + \ln(e^t+1)e^{-t} \\ &= -e^{-t} + e^{-2t}\ln(e^t+1) + e^{-t}\ln(e^t+1). \end{aligned}$$

Step 3: The general solution. Combining the complementary solution and the particular solution,

$$\begin{aligned} y_g(t) &= y_c(t) + y_p(t) \\ &= c_1 e^{-2t} + c_2 e^{-t} - e^{-t} + e^{-2t}\ln(e^t+1) + e^{-t}\ln(e^t+1) \\ &= \boxed{c_1 e^{-2t} + c_2^* e^{-t} + \left(e^{-2t} + e^{-t}\right)\ln(e^t+1)}, \quad c_2^* = c_2 - 1. \end{aligned}$$

5. Solve the differential equation

$$(1-x)y'' + xy' + 2y = 0$$

by means of a power series around the point $x_0 = 0$.

(a) (10 points) Find the recurrence relation.

(b) (10 points) Find the first three nonzero terms in each of the linearly independent solution. Then, write the general solution.

We look for a solution in the form of a power series about $x_0 = 0$

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Differentiating it term by term, we obtain

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

(a) Substituting the series for y, y' and y'' in the equation gives

$$(1-x)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=1}^{\infty} n c_n x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0.$$

We shift the index of the first and the second summations to obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty}(n+1)n c_{n+1}x^n + \sum_{n=1}^{\infty} n c_n x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0.$$

Next, we separate the first terms of the first and the fourth series to write

$$2c_2 + \sum_{n=1}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty}(n+1)n c_{n+1}x^n + \sum_{n=1}^{\infty} n c_n x^n + 2c_0 + 2\sum_{n=1}^{\infty} c_n x^n = 0$$

and therefore to combine the series as

$$(2c_2 + 2c_0) + \sum_{n=1}^{\infty}\left[(n+2)(n+1)c_{n+2} - n(n+1)c_{n+1} + (n+2)c_n\right]x^n = 0.$$

In order for the above equation to be satisfied, we must have

$$2c_2 + 2c_0 = 0, \quad (n+2)(n+1)c_{n+2} - n(n+1)c_{n+1} + (n+2)c_n = 0,$$

which yields the recurrence relation

$$c_2 = -c_0, \quad c_{n+2} = \frac{n(n+1)c_{n+1} - (n+2)c_n}{(n+2)(n+1)}, \quad n = 1, 2, \ldots$$

or

$$\boxed{c_{n+2} = \frac{n}{n+2}c_{n+1} - \frac{1}{n+1}c_n, \quad n = 1, 2, \ldots}$$

(b) From the recurrence relation together with $c_2 = -c_0$, we get

$$\begin{aligned} n = 1 \quad &\Rightarrow \quad c_3 = \frac{c_2}{3} - \frac{c_1}{2} = -\frac{c_0}{3} - \frac{c_1}{2}, \\ n = 2 \quad &\Rightarrow \quad c_4 = \frac{c_3}{2} - \frac{c_2}{3} = \frac{1}{2}\left(-\frac{c_0}{3} - \frac{c_1}{2}\right) + \frac{c_0}{3} = \frac{c_0}{6} - \frac{c_1}{4}, \end{aligned}$$

which is sufficient to determine at least first three nonzero terms in each of the linerly independent solution.

Then, we express the general solution as

$$\begin{aligned}
y_g(x) &= c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots \\
&= c_0 + c_1 x - c_0 x^2 + \left(-\frac{c_0}{3} - \frac{c_1}{2}\right) x^3 + \left(\frac{c_0}{6} - \frac{c_1}{4}\right) x^4 + \cdots + c_n x^n + \cdots \\
&= \boxed{c_0 \left(1 - x^2 - \frac{x^3}{3} + \cdots\right) + c_1 \left(x - \frac{x^3}{2} - \frac{x^4}{4} + \cdots\right)}.
\end{aligned}$$