MATH 255 Differential Equations
Final Exam / 13.01.2026

Solution Key

1	2	3	4	5	6	Total
2+9+4	10+10	10+10	3+7+7+3	10+5	20	110

Read the instructions below.

- *This exam has 6 questions (verify).*
- *The 5th question is from WebWork. You are expected to provide detailed calculations and explanations that clarify the method of your solution, including those from WebWork. If the method is not specified in the question, you may use any suitable approach for solving the problem.*
- *A correct answer without any calculations/explanations will not receive any credit. An incorrect answer, including partially correct calculations, will receive partial credit.*
- *You may not use any electronic device such as phones, smart watches, etc., including calculators. Attempting to use any electronic device will be treated as a cheating attempt.*
- *You may not leave the exam **during the first 30 minutes.***

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1) *Consider the following differential equation*

$$xe^{2y}y' - \frac{e^{2y}}{2} = \frac{2-x}{2x}, \quad x > 1.$$

a) *Answer the following:*

i. *Is the differential equation linear or nonlinear?* Nonlinear.

ii. *What is the order of the differential equation?* One.

b) *Find the general solution. (Hint: Use the substitution $z(x) = e^{2y(x)}$).*

Let $z = e^{2y}$, then $z' = 2e^{2y}y'$. Using the substitution, the differential equation transforms into the following first-order, linear equation

$$xz' - z = \frac{2-x}{x}.$$

It can be written in standard form as

$$z' - \frac{1}{x}z = \frac{2}{x^2} - \frac{1}{x},$$

so we find the integration factor $\mu(x) = e^{-\int (1/x)dx} = \frac{1}{x}$. Next, we multiply both sides by $\mu(x)$ to obtain

$$\frac{d}{dx}\left(\frac{z(x)}{x}\right) = \frac{2}{x^3} - \frac{1}{x^2}$$

and then integrate both sides of the resulting equation to get

$$\frac{z(x)}{x} = -\frac{1}{x^2} + \frac{1}{x} + C \quad \Rightarrow \quad z(x) = -\frac{1}{x} + 1 + Cx.$$

Using the substitution $z = e^{2y}$ back, we obtain the general solution as

$$y_g(x) = \frac{1}{2}\ln\left|-\frac{1}{x} + 1 + Cx\right|.$$

c) *Can you find an initial condition imposed at $x = 1$ so that the solution remains finite as $x \to \infty$?*

Observe that for $C \neq 0$, the limit $\lim_{x\to\infty} y_g(x)$ does not finite. Therefore, finite limit is guaranteed only when $C = 0$. However, for $C = 0$, associated solution becomes

$$y(x) = \frac{1}{2}\ln\left|-\frac{1}{x} + 1\right|$$

which is not defined at $x = 1$. Therefore, there is no initial condition that guarantees the solution remains finite as $x \to \infty$.

2) *Consider the forced second-order linear oscillator*

$$x''(t) + bx'(t) + x(t) = 2\sin(t), \qquad t \geq 0.$$

subject to the initial conditions

$$x(0) = 0, x'(0) = 1.$$

a) *Find the solution for the undamped case ($b = 0$). Describe how the solution behaves as time t increases.*

Step 1 – Complementary solution: For $b = 0$, characteristic equation is $r^2 + 1 = 0$ wşth roots $r_{1,2} = \mp i$. Hence, the complementary solution is

$$x_c(t) = c_1 \sin t + c_2 \cos t.$$

Step 2 – Particular solution and general solution: Using the method of undetermined coefficients, our usual guess, $A\cos t + B\sin t$, *is linearly independent with the complementary solution. Therefore, we multiply it by* t and try the following one

$$x_p(t) = At\cos t + Bt\sin t.$$

Computing the derivatives x_p', x_p'' and substituting into the nonhomogeneous equation $x'' + x = 2\sin t$, we find

$$A = -1, \qquad B = 0.$$

So a particular solution is $x_p(t) = -t\cos t$ and the general solution is

$$x_g(t) = c_1 \sin t + c_2 \cos t - t\cos t.$$

Step 3 – Initial conditions: $x_g(0) = 0 \Rightarrow c_2 = 0$. Therefore, $x(t) = c_1 \sin t - t\cos t$. Computing the derivative $x'(t)$, we find

$$x'(t) = c_1 \cos t - \cos t - t\cos t\,.$$

For $t = 0$, *we have* $x'(0) = 1 \Rightarrow c_1 - 1 = 1 \Rightarrow c_1 = 2$. Hence, solution to the initial value problem is

$$x(t) = 2\sin t - t\cos t.$$

Solution grows unboundedly to $-\infty$ as t increases.

b) *Find the solution for the damped case with $b = 1$. Describe how the solution behaves as $t \to \infty$.*

Following the same steps in part (a), we find the complementary solution as

$$x_c(t) = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

and particular solution as

$$x_p(t) = -2\cos t.$$

Hence, the general solution is

$$x_g(t) = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right) - 2\cos t.$$

Note that no matter what c_1 *and* c_2 *is, complementary part tends to zero as* $t \to \infty$*, therefore*

$$x_g(t) \sim -2\cos t\,, \quad as\ t \to \infty.$$

Now employing the same initial conditions and after calculations, we find $c_1 = 2$ *and* $c_2 = \frac{4}{\sqrt{3}}$*. Hence, solution to the initial value problem is*

$$x_g(t) = e^{-\frac{t}{2}}\left(2\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{4}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right) - 2\cos t.$$

3) *A differential equation with variable coefficients is given as*

$$(1-x)y''+xy'-y=g(x), \qquad 0<x<1.$$

a) *For $g(x)=0$, verify that the homogeneous equation has a solution $y_1(x)=e^x$ and find a second linearly independent solution.*

Substituting $y_1(x)=e^x$ and its derivatives y_1', y_1'' into the homogeneous equation, we get

$$(1-x)e^x+xe^x-e^x=e^x(1-x+x-1)=0, \quad \text{for all } x\in(0,1).$$

Hence, e^x is a solution to the homogeneous equation. To find a second linearly independent solution, we can apply method of reduction of order. To this end, let us look for a second linearly independent solution y_2 in the form $y_2(x)=v(x)e^x$. Calculating the derivatives,

$$y_2'(t)=e^x\big(v(x)+v'(x)\big), \qquad y_2''(x)=e^t\big(v(x)+2v'(x)+v''(x)\big)$$

and then subtituting y_2, y_2', y_2'' into the homogeneous equation yields the following differential equation with the unknown v.

$$(1-x)v''+(2-x)v'=0$$

Set $w=v'$, then the above equation transforms into the following separable equation

$$(1-x)w'+(2-x)w=0 \Rightarrow \frac{w'}{w}=-\frac{2-x}{1-x}.$$

Integrating both sides yields

$$\ln w=-\int\frac{2-x}{1-x}dx=-\int\left(1+\frac{1}{1-x}\right)dx=-x+\ln|1-x|=-x+\ln(1-x),$$

since $0<x<1$. Then,

$$w(x)=(1-x)e^{-x}.$$

To find $v(x)$, which is related via $v'=w$, we integrate (by using integration by parts – Calculus I) with respect to x and obtain

$$v(x)=\int w'(x)dx=\int(1-x)e^{-x}dx=xe^{-x}.$$

Hence, we find the second liearly independent solution as

$$y_2(x)=v(x)y_1(x)=x.$$

b) *Find the general solution of the nonhomogeneous equation when*

$$g(x) = 2(x-1)^2 e^{-x}.$$

Having the fundamental solutions $y_1(x) = e^x$, $y_2(x) = x$ leads us to apply variation of parameters to find a particular solution. To this end, let us first put the nonhomogeneous equation in standard form

$$y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = -2(x-1)e^{-x}.$$

Fundamental solutions are $y_1(x) = e^x$, $y_2(x) = x$ so the Wronskian is

$$W\big(y_1(x), y_2(x)\big) = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x(1-x).$$

Then, in view of the variation of parameters method, particular solution is given by $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ *in which*

$$u_1(x) = 2\int \frac{x(x-1)e^{-x}}{e^x(1-x)}dx = -2\int xe^{-2x}dx = \frac{(2x+1)e^{-2x}}{2}$$

and

$$u_2(x) = -2\int \frac{e^x(x-1)e^{-x}}{e^x(1-x)}dt = 2\int e^{-x}dx = -2e^{-x}.$$

Hence, particular solution is

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = \frac{(2x+1)e^{-x}}{2} - 2xe^{-x} = \left(\frac{1}{2} - x\right)e^{-x}$$

and *the general solution is*

$$y_g(x) = c_1 e^x + c_2 x + \left(\frac{1}{2} - x\right)e^{-x}.$$

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4) *Consider the initial value problem.*

$$y'' + 4y' + 5y = g(t), \quad y(0) = 0, \quad y'(0) = 1,$$

where

$$g(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t \end{cases}$$

a) *Write $g(t)$ in terms of the unit step function $u_c(t)$.*

$g(t) = t - u_1(t)(t-1).$

b) *Apply the Laplace transform to both sides of the ODE. Solve for $Y(s) = L\{y(t)\}$.*

Taking into account the initial conditios, Laplace transform of left hand side is

$$\mathcal{L}\{y'' + 4y' + 5\} = s^2Y(s) + 4sY(s) + 5Y(s) - 1.$$

Using properties 3 and 13, right hand side becomes

$$\mathcal{L}\{t - u_1(t)(t-1)\} = \frac{1}{s^2} - \frac{e^{-s}}{s^2}.$$

Consequently,

$$s^2Y(s) + 4sY(s) + 5Y(s) - 1 = \frac{1}{s^2} - \frac{e^{-s}}{s^2} \quad \Rightarrow \quad Y(s) = \frac{1}{s^2+4s+5} + \frac{1}{(s^2+4s+5)s^2} - \frac{e^{-s}}{(s^2+4s+5)s^2}.$$

c) *Find $y(t)$ by computing the inverse Laplace transform of $Y(s)$.*

Denote

$$Y_1(s) = \frac{1}{s^2+4s+5}, \qquad Y_2(s) = \frac{1}{(s^2+4s+5)s^2}, \qquad Y_3(s) = \frac{e^{-s}}{(s^2+4s+5)s^2} = e^{-s}Y_2(s)$$

and $y_1(t) = \mathcal{L}^{-1}\{Y_1(s)\}$, $y_2(t) = \mathcal{L}^{-1}\{Y_2(s)\}$, $y_3(t) = \mathcal{L}^{-1}\{Y_3(s)\}$.

- Using the property 9,

$$y_1(t) = \mathcal{L}^{-1}\{Y_1(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\sin t.$$

- Using partial fraction decomposition (Calculus I), we can write $Y_2(s)$ as

$$Y_2(s) = \frac{1}{(s^2+4s+5)s^2} = -\frac{4}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2} + \frac{1}{25}\frac{4s+11}{s^2+4s+5}$$

or in a more convenient way

$$Y_2(s) = -\frac{4}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2} + \frac{1}{25}\left(4 \times \frac{s+2}{(s+2)^2+1} + 3 \times \frac{1}{(s+2)^2+1}\right)$$

so that properties 9 and 10 are applicable for the last two term. Thus,

$$y_2(t) = \mathcal{L}^{-1}\{Y_2(s)\} = \mathcal{L}^{-1}\left\{-\frac{4}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2} + \frac{1}{25}\left(4 \times \frac{s+2}{(s+2)^2+1} + 3 \times \frac{1}{(s+2)^2+1}\right)\right\}$$

$$= -\frac{4}{25} + \frac{t}{5} + \frac{4}{25}e^{-2t}\cos t + \frac{3}{25}e^{-2t}\sin t.$$

- *Using Property 13*

$$y_3(t) = \mathcal{L}^{-1}\{e^{-s}Y_2(s)\} = u_1(t)y_2(t-1).$$

Hence, the solution to the initial value problem

$$y(t) = y_1(t) + y_2(t) - y_3(t),$$

where

$$y_1(t) = e^{-2t}\sin t, \qquad y_2(t) = -\frac{4}{25} + \frac{t}{5} + \frac{4}{25}e^{-2t}\cos t + \frac{3}{25}e^{-2t}\sin t, \qquad y_3(t) = u_1(t)y_2(t-1).$$

d) *Describe the behavior of $y(t)$ as $t \to \infty$.*

Solution to the initial value problem consists of three type of fuctions: Constant functions, polynomials with first-degree, exponential functions with a negative exponent multiplied by sines and cosines. The constant function remains constant no matter what t is. Due to the negative exponent, the exponentials multiplied by sines and cosines tend to zero as $t \to \infty$. However, the polynomial t will grow unboundedly as $t \to \infty$. Hence $\lim_{t\to\infty} y(t) = \infty$.

5) (WebWork) *Consider the linear system*

$$y' = \begin{bmatrix} -6 & -4 \\ 12 & 8 \end{bmatrix} y.$$

a) *Find the eigenvalues and eigenvectors for the coefficient matrix.*

$$\lambda_1 = \_\_\_\_, \quad v_1 = \begin{bmatrix} \\ \\ \end{bmatrix}, \quad \text{and } \lambda_2 = \_\_\_\_, \quad v_2 = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$\det(A - \lambda I) = det \begin{bmatrix} -6-\lambda & -4 \\ 12 & 8-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda = 0.$$

Hence the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$. *Let* $v_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ and $v_2 = \begin{bmatrix} c \\ d \end{bmatrix}$ be the associated eigenvectors.

- $\lambda_1 = 0$:

$$(A - \lambda_1 I)v_1 = 0 \Rightarrow \begin{bmatrix} -6 & -4 \\ 12 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow 3a + 2b = 0.$$

Choose $a = 2$, $b = -3$, then $v_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

- $\lambda_2 = 2$:

$$(A - \lambda_2 I)v_2 = 0 \Rightarrow \begin{bmatrix} -8 & -4 \\ 12 & 6 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = 0 \Rightarrow 2c + d = 0.$$

Choosing $c = 1$, $d = -2$ yields $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

b) *For each eigenpair in the previous part, form a solution of* $y' = Ay$. *Use t as the independent variable in your answers.*

- Solution associated to the first eigenpair (λ_1, v_1) is $v_1 e^{\lambda_1 t} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.
- Solution associated to the second eigenpair (λ_2, v_2) is $v_2 e^{\lambda_2 t} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{2t}$.

6) *Solve the following system of first-order equations*

$$x' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} x.$$

Step 1 – Eigenvalues:

$$\det(A - \lambda I) = det \begin{bmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{bmatrix} = 0 \Rightarrow (\lambda+1)^2 + 4 = 0 \Rightarrow \lambda_{1,2} = -1 \mp 2i.$$

Step 2 – A complex eigenvector: Choose $\lambda_1 = -1 + 2i$. Then,

$$A - \lambda_1 I = \begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix}.$$

Let $v = \begin{bmatrix} a \\ b \end{bmatrix}$. Solving the system $(A - \lambda_1 I)v = 0$ yields

$$-2ia - 4b = 0 \Rightarrow a = 2ib.$$

Take $b = 1$, then $a = -2i$ and the associated eigenvector is

$$v = \begin{bmatrix} 2i \\ 1 \end{bmatrix}.$$

Step 3 – Real-valued general solution: The eigenpair we found above yields the following complex-valued solution

$$ve^{\lambda_1 t} = \begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{(-1+2i)t}.$$

To find two linearly independent real-valued solution, let us decompose $ve^{\lambda_1 t}$ into its real and imaginary parts.

$$\begin{aligned} ve^{\lambda_1 t} &= \begin{bmatrix} 2i \\ 1 \end{bmatrix} e^{(-1+2i)t} = e^{-t} \begin{bmatrix} 2i \\ 1 \end{bmatrix} (\cos 2t + i \sin 2t) = e^{-t} \begin{bmatrix} -2\sin 2t + 2i \cos 2t \\ \cos 2t + i \sin 2t \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} -2\sin 2t \\ \cos 2t \end{bmatrix} + ie^{-t} \begin{bmatrix} 2\cos 2t \\ \sin 2t \end{bmatrix}. \end{aligned}$$

Hence,

$$e^{-t} \begin{bmatrix} -2\sin 2t \\ \cos 2t \end{bmatrix} \quad \text{and} \quad e^{-t} \begin{bmatrix} 2\cos 2t \\ \sin 2t \end{bmatrix}$$

are linearly independent two solutions and the real-valued general solution is

$$x_g(t) = c_1 e^{-t} \begin{bmatrix} -2\sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2\cos 2t \\ \sin 2t \end{bmatrix}.$$

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TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\frac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\frac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. t^n; $n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 6
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \|a\|$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \|a\|$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}$, $n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)\, f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct} f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 19
16. $\int_0^t f(t-\tau)g(\tau)\, d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28

Duration: 13:30 – 16:00