MATH 255 Differential Equations
Midterm 1 / 30.10.2025

Solution Key

1	2	3	4	5	Total
25	20	20	25	20	110

1) *Find the solution of the following first order linear initial value problem,*

$$\frac{dy}{dt} + 2y = e^{-2t+\cos(t)}\sin(t), \qquad y(0) = 1 - e\,.$$

Where does the solution go for $t \to \infty$? Is it possible to change the long time behavior changing the initial condition?

The equation is first-order and linear. The integration factor is

$$\mu(t) = e^{\int 2dt} = e^{2t}.$$

Multiplying both sides of the equation by $\mu(t) = e^{2t}$ yields

$$e^{2t}\frac{dy}{dt} + 2e^{2t}y = e^{\cos(t)}\sin(t) \Rightarrow \frac{d}{dt}(e^{2t}y) = e^{\cos(t)}\sin(t).$$

We integrate both sides to obtain

$$e^{2t}y = \int e^{\cos(t)}\sin(t)\,dt \Rightarrow e^{2t}y = -e^{\cos(t)} + C.$$

Hence, the general solution is explicitly

$$y(t) = -e^{-2t+\cos(t)} + Ce^{-2t}.$$

To determine C, let us impose the given initial condition and take $t = 0$

$$t = 0 \Rightarrow y(0) = -e^{\cos(0)} + C = 1 - e \Rightarrow C = 1.$$

Hence, solution of the given initial value problem is

$$y(t) = -e^{-2t+\cos(t)} + e^{-2t}$$

and the solution goes to $y(t) \to 0$ as $t \to \infty$. Looking at the general solution

$$y(t) = -e^{-2t+\cos(t)} + Ce^{-2t}$$

the asymptotic property $\lim_{t\to\infty} y(t) = 0$ is true no matter what C is. Hence, it is not possible to change long time behavior of solutions by changing the initial condition.

Duration: 13:30 – 15:15

2) ***(WebWork)*** *Use an appropriate substitution to solve the equation,*

$$x\frac{dy}{dx} + y = 2xy^2, \qquad x > 0,$$

and find the solution that satisfies $y(1) = 4$.

Given equation is Bernoulli type differential equation with $\mathrm{n} = 2$. Therefore, the following substitution

$$z = y^{1-2} = y^{-1} \Rightarrow z' = -y^{-2}y'$$

transforms the Bernoulli equation into the following linear equation

$$-xz' + z = 2x \Rightarrow z' - x^{-1}z = -2.$$

To solve the z $-$equation, first we multiply both sides by the associated integration factor

$$\mu(x) = e^{\int\left(-\frac{1}{x}\right)dx} = x^{-1}$$

and write

$$x^{-1}z' - x^{-2}z = -2x^{-1} \Rightarrow \frac{d}{dx}(x^{-1}z) = -2x^{-1}.$$

Then, we integrate both sides to obtain

$$x^{-1}z = -2\ln x + C \Rightarrow z(x) = -2x\ln x + Cx.$$

Finally, we substitute $z = y^{-1}$ back to obtain the general solution of y $-$*equation as*

$$\frac{1}{y(x)} = -2x\ln x + Cx.$$

To determine C*, we take* $x = 1$ *and impose* $y(1) = 4$*. This yields*

$$\frac{1}{4} = C.$$

Hence, solution of the initial value problem is given by

$$\frac{1}{y(x)} = -2x\ln x + \frac{x}{4}.$$

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3) *Consider the following differential equation*

$$\frac{dy}{dx} = \frac{4y - 3x}{2x - y}.$$

a) *Show that the right hand side of the differential equation is homogeneous.*

b) *Solve the differential equation.*

a) *The right hand side can be expressed as*

$$\frac{4y - 3x}{2x - y} = \frac{4\left(\frac{y}{x}\right) - 3}{2 - \left(\frac{y}{x}\right)}$$

which is a function of $\frac{y}{x}$. Hence, it is a homogeneous function.

b) *Set $y = ux$, then $y' = u + xu'$. We use this substitution and the resulting differential equation becomes*

$$u + xu' = \frac{4u - 3}{2 - u}.$$

or equivalently

$$\frac{1}{x} + \frac{u - 2}{u^2 + 2u - 3} u' = 0.$$

Observe that the equation is separable. Integrating term by term yields

$$\int \frac{1}{x} dx = \ln|x|, \qquad \int \frac{u(x) - 2}{u^2(x) + 2u(x) - 3} u'(x) dx = \frac{5}{4}\ln|u(x) + 3| - \frac{1}{4}\ln|u(x) - 1|$$

(the second one requires partial fraction decomposition). Hence, the general solution of the u —equation is given by

$$\ln|x| + \frac{5}{4}\ln|u(x) + 3| - \frac{1}{4}\ln|u(x) - 1| = C.$$

Finally, we substitute back $u = \frac{y}{x}$ and obtain the general solution as

$$\ln|x| + \frac{5}{4}\ln\left|\frac{y(x)}{x} + 3\right| - \frac{1}{4}\ln\left|\frac{y(x)}{x} - 1\right| = C.$$

4) *Consider a large tank holding 1000 L of pure water into which a brine solution of salt begins to flow at a constant rate of $6L/min$. The solution inside the tank is kept well stirred and is flowing out of the tank at a rate of 6 L/min. The concentration of salt in the brine entering the tank is $0.1\ kg/L$.*

a) *Determine the mass of the salt in the tank after t min.*

b) *Show that limiting concentration of salt is $0.1\ kg/L$.*

Let t denotes the time in minutes and $y(t)$ denotes the total amount of salt in kilograms.

a) *Input rate: The brine flows into the tank at a rate of 6 L/min with concentration 0.1 kg/L. Therefore, the input rate of salt into the tank is*

$$6 \times 0.1 = 0.6\ kg/min.$$

Output rate: The tank initially contains 1000 L and the rate of flow into the tank is the same as the rate of flow out which is 6 L/min. Therefore, the volume remains constant 1000 L and the concentration of salt for all time is just $y(t)$ divided by 1000 L. So, the output rate of salt is

$$6 \times \frac{y(t)}{1000} = \frac{3y(t)}{500} kg/min.$$

The tank initially contains pure water, so we set $y(0) = 0$. Consequently, the governing initial value problem that describes the evolution of amount of salt (measured in kilograms) in time (measured in minutes) is given by

$$y' = 0.6 - \frac{3y}{500}, \quad y(0) = 0.$$

This is a first-order, linear equation. Solving this initial value problem by the method of integration factors yields

$$y(t) = 100\left(1 - e^{-\frac{3t}{500}}\right),$$

which is the function that describes the amount of salt in the tank at time t.

b) *Since the volume in the tank is constant with 1000 L, the concentration of salt in the tank is given by*

$$\frac{y(t)}{1000} = 0.1\left(1 - e^{-\frac{3t}{500}}\right).$$

Taking limit as $t \to \infty$, we find that $\lim_{t\to\infty} \frac{y(t)}{1000} = 0.1$.

5) *Find the solution of the following initial value problem,*

$$(ye^{xy} + \cos x)dx + xe^{xy}dy = 0, \qquad y\left(\frac{\pi}{2}\right) = 0.$$

Denote $M(x,y) = ye^{xy} + \cos x$ *and* $N(x,y) = xe^{xy}$*. Then,*

$$\left.\begin{aligned} \frac{\partial}{\partial y}M(x,y) &= \frac{\partial}{\partial y}(ye^{xy} + \cos x) = e^{xy} + xye^{xy} \\ \frac{\partial}{\partial x}N(x,y) &= \frac{\partial}{\partial x}(xe^{xy}) = e^{xy} + xye^{xy} \end{aligned}\right\} \Rightarrow \frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y).$$

Therefore, the equation is exact and there exists a function $\Psi(x,y)$ *such that* $\Psi_x(x,y) = M(x,y)$ *and* $\Psi_y(x,y) = N(x,y)$*. To find the most general form of* $\Psi(x,y)$*, let us integrate* $M(x,y)$ *with respect to* x *while holding* y *as a constant:*

$$\Psi(x,y) = \int M(x,y)dx = \int (\, ye^{xy} + \cos x\,)dx = e^{xy} + \sin x + g(y).$$

To determine $g(y)$*, we differentiate* $\Psi(x,y) = e^{xy} + \sin x + g(y)$ *with respect to* y *and set the result equal to* $N(x,y)$*:*

$$\frac{\partial}{\partial y}\big(e^{xy} + \sin x + g(y)\big) = xe^{xy} \Rightarrow g'(y) = 0 \Rightarrow g(y) = C.$$

Thus, the general solution is given implicitly by

$$e^{xy} + \sin x + C = 0.$$

Finally, we take $x = \frac{\pi}{2}$ *and use* $y\left(\frac{\pi}{2}\right) = 0$ *to obtain* $C = -2$*. Hence, solution of the initial value problem is given by*

$$e^{xy} + \sin x - 2 = 0.$$