MATH 255 Differential Equations
Midterm 3 / 25.12.2025

Solution Key

1	*2*	*3*	*4*	*5*	*Total*
20	**25**	**10+10**	**25**	**10+10**	**110**

1) *Use the method of reduction of order to find a second solution of the given differential equation*

$$t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0; \quad y_1(t) = t.$$

Let y_2 be a second linearly independent solution and set $y_2(t) = tv(t)$. We differentiate y_2 up to the order two and get

$$y_2'(t) = v(t) + tv'(t), \qquad y_2''(t) = 2v'(t) + tv''(t).$$

Substituting y_2, y_2' and y_2'' into the equation and then simplifying the result, we obtain

$$v'' - v' = 0.$$

Set $v' = w$, then the v −equation reduces to $w' - w = 0$ which has a solution $w(t) = e^t$. By integrating, we find $v(t) = e^t$. Hence a second linearly independent solution to the given differential equation is

$$y_2(t) = tv(t) = te^t.$$

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2) *Find the power series solution of the following differential equation in terms of the powers of x, i.e., about $x_0 = 0$. (Determine the recurrence relation, find the first 3 terms for each linearly independent solution.)*

$$(2+x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$$

We look for a solution in the form of a power series about $x_0 = 0$

$$y(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Differentiating term by term, we obtain

$$y'(x) = \sum_{k=1}^{\infty} c_k k x^{k-1}, \qquad y(x) = \sum_{k=2}^{\infty} c_k k(k-1) x^{k-2}.$$

Substituting y, y' and y'' in the equation yields

$$(2+x^2)\sum_{k=2}^{\infty} c_k k(k-1)x^{k-2} - x\sum_{k=1}^{\infty} c_k k x^{k-1} + 4\sum_{k=0}^{\infty} c_k x^k = 0$$

or

$$2\sum_{k=2}^{\infty} c_k k(k-1)x^{k-2} + \sum_{k=2}^{\infty} c_k k(k-1)x^k - \sum_{k=1}^{\infty} c_k k x^k + 4\sum_{k=0}^{\infty} c_k x^k = 0.$$

We shift the index of the first series by 2 and write

$$2\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=2}^{\infty} c_k k(k-1)x^k - \sum_{k=1}^{\infty} c_k k x^k + 4\sum_{k=0}^{\infty} c_k x^k = 0.$$

Setting the index of the second and the third series to start from $k = 0$, we can write

$$2\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=0}^{\infty} c_k k(k-1)x^k - \sum_{k=0}^{\infty} c_k k x^k + 4\sum_{k=0}^{\infty} c_k x^k = 0.$$

Thus, we can combine the left hand side of the above equation in powers of x as following

$$\sum_{k=0}^{\infty}[2(k+2)(k+1)c_{k+2} + k(k-1)c_k - kc_k + 4c_k]x^k = 0.$$

In order for the above equation holds, we must have

$$c_{k+2} = -\frac{k^2-2k+4}{2(k+2)(k+1)}c_k,$$

where k is a nonnegative integer. From the recurrence relation, we find

$$k = 0 \Rightarrow c_2 = -c_0,$$

$$k = 1 \Rightarrow c_3 = -\frac{1}{4}c_1,$$

$$k = 2 \Rightarrow c_4 = -\frac{1}{6}c_2 = \frac{c_0}{6},$$

$$k = 3 \Rightarrow c_5 = -\frac{7}{40}c_3 = \frac{7c_1}{160}.$$

Hence the general solution is

$$y_g(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k + \cdots = c_0 + c_1 x - c_0 x^2 - \frac{c_1}{4}x^3 + \frac{c_0}{6}x^4 + \frac{7c_1}{160}x^5 + \cdots + c_k x^k + \cdots$$

$$= c_0\left(1 - x^2 + \frac{1}{6}x^4 + \cdots\right) + c_1\left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5 + \cdots\right).$$

3) a) *Suppose that* $y_1(x) = \frac{\ln x}{x}$ *is a solution to the following differential equation*

$$x^2y'' + axy' + by = 0, \quad x > 0.$$

Find a second linearly independent solution and determine the coefficients a, b.

Given differential equation is a Cauchy-Euler equation. Note that $\ln x$ that appears in the solution multiplied by x^{-1} indicates that $r = -1$ is the repeated root of the associated characteristic equation. Therefore, we infer from this observation that, given one solution

$$y_1(x) = x^{-1}\ln x,$$

a second linearly independent solution is

$$y_2(x) = x^{-1}.$$

Note also that associated characteristic equation with a repeated root $r = -1$ is $r^2 + 2r + 1 = 0$. This implies $a = 3$ and $b = 1$.

Note: We know that the substitution $t = \ln x$ (or $x = e^t$) transforms a Cauchy-Euler equation to a constant coefficient homogeneus equation. So, alternatively we can find y_2 and determine the coefficients a,b by studying the transformed equation.

To this end, by using the substitution we can write y_1 in the variable t as

$$y_1(t) = te^{-t}.$$

Observing that $y_1(t)$ is a product of e^{-t} with t, we infer that a second linearly independent solution in the variable t is $y_2(t) = e^{-t}$. Using the relation $x = e^t$ back, we find y_2 in the variable x as

$$y_2(x) = x^{-1}.$$

b) *Consider the Cauchy-Euler equation*

$$x^2y'' + \beta xy' = 0, \quad x > 1.$$

(i) *Let* $\beta = 1$. *Is there a pair of initial conditions,* $y(1) = y_0$, $y'(1) = y_1$, *that guarantees some solutions remain bounded as* $x \to \infty$? *If your answer is yes, find one. If your answer is no, explain why it is not possible.*

For $\beta = 1$, characteristic equation is $r^2 - r + r = 0 \Rightarrow r^2 = 0$. Therefore, we have a repeated real root $r_{1,2} = 0$ and the general solution is given by

$$y(x) = c_1 + c_2\ln x, \quad c_1, c_2 \in \mathbb{R}.$$

The unbounded term in this form is the logaritmic term. Therefore, in order to have a solution which remains bounded as $x \to \infty$, we must have $c_2 = 0$. Then, the solution becomes a constant. That is, in particular, a solution with a zero slope at $x = 1$. Hence, an asymptotically bounded solution is achieved if the second initial condition is prescribed as

$$y'(1) = 0.$$

(ii) *Find a condition on β so that, no matter what initial conditions are prescribed, all solutions remain bounded as $x \to \infty$.*

For a general β, characteristic equation is

$$r^2 - r + \beta r = 0 \Rightarrow r(r - 1 + \beta) = 0$$

with roots $r_1 = 0$, $r_2 = 1 - \beta$. Note that for $\beta = 1$, roots become repeated which is already investigated in part (i) and concluded that not all solutions remain bounded as $x \to \infty$. For $\beta \neq 1$, the general solution is

$$y(x) = c_1 + c_2 x^{1-\beta}.$$

Depending on the sign of the power in the second term, we have two cases:

- If $\beta > 1$, *then* $1 - \beta < 0$. *For such* β, no matter what c_1 and c_2 is, the limit $\lim_{x\to\infty} y(x)$ remains finite.
- Conversely if $\beta < 1$, then there are some solutions which grow unboundedly as $x \to \infty$.

Hence, the answer is $\beta > 1$.

4) *Find the solution of the following initial value problem* ***using the Laplace transformation method.***

$$y'' + 4y = 25te^t, \quad y(0) = 1, \quad y'(0) = -1.$$

Denote $\mathcal{L}\{y(t)\} = Y(s)$ and take Laplace transform of both sides of the equation:

$$\mathcal{L}\{y''(t) + 4y(t)\} = \mathcal{L}\{25te^t\}.$$

Using the linearity of Laplace transform and taking into account the initial conditions, left hand side becomes

$$\mathcal{L}\{y''(t) + 4y(t)\} = \mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} = s^2Y(s) - s + 1 + 4Y(s).$$

Using the property 11, right hand side is

$$\mathcal{L}\{25te^t\} = \frac{25}{(s-1)^2}.$$

Consequently, from the transformed equation, we find $Y(s)$ as

$$s^2Y(s) - s + 1 + 4Y(s) = \frac{25}{(s-1)^2} \Rightarrow Y(s) = \frac{25}{(s^2+4)(s-1)^2} + \frac{s-1}{s^2+4}.$$

The first term on the right hand side can be decomposed into partial fractions as

$$\frac{25}{(s^2+4)(s-1)^2} = \frac{2s-3}{s^2+4} - \frac{2}{s-1} + \frac{5}{(s-1)^2}$$

Thus, $Y(s)$ can be reexpressed as

$$Y(s) = \frac{3s-4}{s^2+4} - \frac{2}{s-1} + \frac{5}{(s-1)^2}$$

or in a more convenient way (convenient in the sense that we can directly use the transform table sheet)

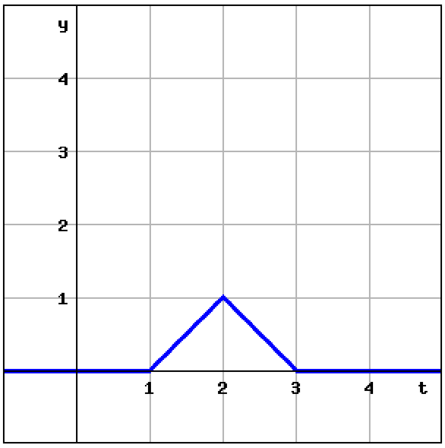
$$Y(s) = 3 \times \frac{s}{s^2+4} - 2 \times \frac{2}{s^2+4} - 2 \times \frac{1}{s-1} + 5 \times \frac{1}{(s-1)^2}.$$

Now using linearity of inverse Laplace transform and the properties 6, 5, 2, 11 in the respective order, we obtain the solution to the given initial value problem as

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\left\{3 \times \frac{s}{s^2+4} - 2 \times \frac{2}{s^2+4} - 2 \times \frac{1}{s-1} + 5 \times \frac{1}{(s-1)^2}\right\} \\ &= 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} \\ &= 3\cos(2t) - 2\sin(2t) - 2e^t + 5te^t. \end{aligned}$$

5) (WebWork)

a) *The graph of $f(t)$ is given in the figure below. Represent $f(t)$ using a combination of unit step functions.*



$f(t) = u_1(t)(t-1) - 2u_2(t)(t-2) + u_3(t)(t-3).$

b) *Find its Laplace transform $F(s) = \mathcal{L}\{f(t)\}$.*

Using the linearity of Laplace transform and properties 3, 12 from the table, we write

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_1(t)(t-1) - 2u_2(t)(t-2) + u_3(t)(t-3)\}$$
$$= \mathcal{L}\{u_1(t)(t-1)\} - 2\mathcal{L}\{u_2(t)(t-2)\} + \mathcal{L}\{u_3(t)(t-3)\} = \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}.$$